Inverse Laplace transforms

Defn: If $L[f(t)] = f(s)$ then $f(t)$ is called the inverse L.T of $f(s)$ and is denoted by $L^{-1}[f(s)]$

thus $L[f(t)] = f(s) \Rightarrow L^{-1}[f(s)] = f(t)$

Ex:- $L[1] = 1/s \Rightarrow L^{-1}[1/s] = 1$

$L[\cos at] = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

~~Formula~~

formula list

function	Inverse transform	function	Inverse transform
1) $\frac{1}{s}$	1	6) $\frac{a}{s^2+a^2}$	$\sin at$
2) $\frac{1}{s-a}$	e^{at}	7) $\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
3) $\frac{1}{s+a}$	e^{-at}	8) $\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
4) $\frac{s}{s^2+a^2}$	$\cos at$	9) $\frac{1}{s^{n+1}}$ $n=1,2,3 \ldots$	$\frac{t^n}{n!}$
5) $\frac{s}{s^2-a^2}$	$\cosh at$		

Examples

1) $L^{-1}\left[\frac{1}{s-1}\right] = e^{t}$

2) $L^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t$

3) $L^{-1}\left[\frac{1}{s^2+5}\right] = \frac{1}{\sqrt{5}} \sin(\sqrt{5}t)$

4) $L^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$

5) $L^{-1}\left[\frac{s}{s^2-16}\right] = \cosh 4t$

6) $L^{-1}\left[\frac{1}{s^2-36}\right]$

$= \frac{1}{6} \sinh 6t$

Linearity property

$$L^{-1}[c_1 f(s) + c_2 g(s)] = c_1 L^{-1}[f(s)] + c_2 L^{-1}[g(s)]$$

Find the inverse Laplace transform of the following functions

1) $\frac{1}{s+2} + \frac{3}{2s+5} - \frac{4}{3s-2}$

soln. Take inverse L.T

$$L^{-1}\left[\frac{1}{s+2}\right] + L^{-1}\left[\frac{3}{2s+5}\right] - L^{-1}\left[\frac{4}{3s-2}\right]$$

$$= L^{-1}\left[\frac{1}{s+2}\right] + L^{-1}\left[\frac{3}{2(s+5/2)}\right] - L^{-1}\left[\frac{4}{3(s-2/3)}\right]$$

$$= L^{-1}\left[\frac{1}{s+2}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s+5/2}\right] - \frac{4}{3}L^{-1}\left[\frac{1}{s-2/3}\right]$$

$$= e^{-2t} + \frac{3}{2}e^{-5/2\,t} - \frac{4}{3}e^{2/3\,t}$$

2) $\frac{s+2}{s^2+36} + \frac{4s-1}{s^2+25}$

soln. $L^{-1}\left[\frac{s+2}{s^2+36}\right] + L^{-1}\left[\frac{4s-1}{s^2+25}\right]$

$$L^{-1}\left[\frac{s}{s^2+6^2}\right] + L^{-1}\left[\frac{2}{s^2+6^2}\right] + 4L^{-1}\left[\frac{s}{s^2+5^2}\right] - L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$\Rightarrow L^{-1}\left[\frac{s}{s^2+6^2}\right] + 2L^{-1}\left[\frac{1}{s^2+6^2}\right] + 4L^{-1}\left[\frac{s}{s^2+5^2}\right] - L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$= \cos 6t + \frac{2}{6}\sin 6t + 4\cos 5t - \frac{1}{5}\sin 5t$$

$\left[L^{-1}\left[\frac{1}{\ldots}\right] = \frac{1}{\ldots}\sin at \right]$

(3) $\dfrac{2s-5}{4s^2+25} + \dfrac{8-6s}{16s^2+9}$

$= \dfrac{2s}{4s^2+25} - \dfrac{5}{4s^2+25} + \dfrac{8}{16s^2+9} - \dfrac{6s}{16s^2+9}$

apply L^{-1}

$\Rightarrow 2L^{-1}\left[\dfrac{s}{4(s^2+25/4)}\right] - L^{-1}\left[\dfrac{5}{4(s^2+25/4)}\right] + 8L^{-1}\left[\dfrac{1}{16(s^2+9/16)}\right] - 6L^{-1}\left[\dfrac{s}{16(s^2+9/16)}\right]$

$= \dfrac{2}{4} L^{-1}\left[\dfrac{s}{s^2+(\frac{5}{2})^2}\right] - \dfrac{5}{4} L^{-1}\left[\dfrac{1}{s^2+(5/2)^2}\right] + \dfrac{8}{16} L^{-1}\left[\dfrac{1}{s^2+(3/4)^2}\right] - \dfrac{6}{16} L^{-1}\left[\dfrac{s}{s^2+(3/4)^2}\right]$

$= \dfrac{1}{2}\cos\left(\dfrac{5}{2}t\right) - \dfrac{5}{4}\dfrac{1}{(5/2)}\sin\left(\dfrac{5}{2}t\right) + \dfrac{1}{2}\cdot\dfrac{1}{(3/4)}\sin\left(\dfrac{3}{4}t\right) - \dfrac{3}{8}\cos\left(\dfrac{3}{4}t\right)$

$= \dfrac{1}{2}\cos\left(\dfrac{5}{2}t\right) - \dfrac{1}{2}\sin\left(\dfrac{5}{2}t\right) + \dfrac{2}{3}\sin\left(\dfrac{3}{4}t\right) - \dfrac{3}{8}\cos\left(\dfrac{3}{4}t\right)$

4) $\dfrac{2s-5}{8s^2-50} + \dfrac{4s}{9-s^2}$

$= \dfrac{2s-5}{2(4s^2-25)} + \dfrac{4s}{9-s^2}$

$= \dfrac{\cancel{2s-5}}{2[(2s+5)\cancel{(2s-5)}]} - \dfrac{4s}{s^2-9}$ $\quad \left(\because (a+b)(a-b) = a^2-b^2\right)$

apply L^{-1}

$= \dfrac{1}{2}L^{-1}\left[\dfrac{1}{2s+5}\right] - 4L^{-1}\left[\dfrac{s}{s^2-3^2}\right]$

$= \dfrac{1}{2}L^{-1}\left[\dfrac{1}{2(s+5/2)}\right] - 4\cosh 3t$

$= \dfrac{1}{4}e^{-\frac{5}{2}t} - 4\cosh 3t$

5) $\dfrac{(s+2)^3}{s^6} = \dfrac{s^3+8+6s^2+12s}{s^6}$ $\qquad (a+b)^3 = a^3+b^3+3a^2b+\dots$

Apply L^{-1}

$$L^{-1}\left[\frac{(s+2)^3}{s^6}\right] = L^{-1}\left[\frac{s^3}{s^6}\right] + L^{-1}\left[\frac{8}{s^6}\right] + L^{-1}\left[\frac{6s^2}{s^6}\right] + L^{-1}\left[\frac{12s}{s^5}\right]$$

$$= L^{-1}\left[\frac{1}{s^3}\right] + 8L^{-1}\left[\frac{1}{s^6}\right] + 6L^{-1}\left[\frac{1}{s^4}\right] + 12L^{-1}\left[\frac{1}{s^5}\right]$$

$$= \frac{t^2}{2!} + 8\cdot\frac{t^5}{5!} + 6\cdot\frac{t^3}{3!} + 12\cdot\frac{t^4}{4!}$$

$\left[\because L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}\right]$

$$\Rightarrow L^{-1}\left[\frac{(s+2)^3}{s^6}\right] = \frac{t^2}{2} + \frac{t^5}{15} + t^3 + \frac{t^4}{2}$$

6) $L^{-1}\left[\dfrac{3(s^2-1)^2}{2s^5}\right] = \dfrac{3}{2}\left[1 - t^2 + \dfrac{t^4}{24}\right]$

7) $L^{-1}\left[\dfrac{s^2-3s+4}{s^3}\right] = 1-3t+2t^2$

Inverse of $e^{-as} f(s)$

wkt, $L[f(t-a)\,u(t-a)] = e^{-as} f(s)$

$\Rightarrow L^{-1}[e^{-as} f(s)] = f(t-a)\,u(t-a)$

working procedure

1) In the given function, we should observe the presence of e^{-as} first and identify the remaining part of the function to be $f(s)$

2) Taking the inverse L.T of $f(s)$, we obtain $f(t)$.

3) The required inverse of $e^{-as} f(s)$ is obtained by replacing 't' by $(t-a)$ in $f(t)$ to be multiplied by the unit step function $u(t-a)$.

Examples

Find the inverse L.T of the following

1) $\dfrac{1+e^{-3s}}{s^2}$

Asoln :- Take Inverse L.T

$$L^{-1}\left[\frac{1+e^{-3s}}{s^2}\right] = L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{e^{-3s}}{s^2}\right] \rightarrow (1)$$

wkt $L^{-1}[1/s^2] = t$

(1) $\Rightarrow$

$$L^{-1}\left[\frac{1+e^{-3s}}{s^2}\right] = t + (t-3)\,u(t-a)$$

$L^{-1}[e^{-as} f(s)]$
$= f(t-a)\,u(t-a)$

 [redacted]

2> $\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}$

Apply I.L.T

$$L^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right] = 3L^{-1}\left[\frac{1}{s^2}\right] + 2L^{-1}\left[\frac{e^{-s}}{s^3}\right] - 3L^{-1}\left[\frac{e^{-2s}}{s}\right] \longrightarrow (1)$$

wkt $L^{-1}\left[1/s^2\right] = t$, $L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2}$, $L^{-1}\left[1/s\right] = 1$

(1) =>

$$\Rightarrow L^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right] = 3t + \cancel{2}\frac{(t-1)^2}{\cancel{2}}u(t-1) - 3(1)u(t-2)$$

$$= 3t + (t-1)^2 u(t-1) - 3u(t-2)$$

3> $\frac{\cosh 2s}{e^{3s}.s^2}$

soln

$$= \frac{\cosh 2s}{e^{3s}.s^2} = \frac{e^{-3s}}{s^2}\left[\frac{e^{2s} + e^{-2s}}{2}\right] = \frac{1}{2}\left[\frac{e^{2s}.e^{-3s} + e^{-2s}.e^{-3s}}{s^2}\right]$$

Apply I.L.T

$$L^{-1}\left[\frac{\cosh 2s}{e^{3s}.s^2}\right] = \frac{1}{2}\left[L^{-1}\left[\frac{e^{-s}}{s^2}\right] + L^{-1}\left[\frac{e^{-5s}}{s^2}\right]\right] \qquad \left(\cosh ax = \frac{e^{ax} + e^{-ax}}{2}\right)$$

$$= \frac{1}{2}\left[(t-1)u(t-1) + (t-5)u(t-5)\right] \qquad L^{-1}\left[\frac{1}{s^2}\right] = t$$

4> $\frac{e^{-\pi s}}{s^2+1} + \frac{s e^{-2\pi s}}{s^2+4}$

apply ILT

$$L^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right] + L^{-1}\left[\frac{s e^{-2\pi s}}{s^2+4}\right].$$

$$= \sin(t-\pi)u(t-\pi) + \cos 2(t-2\pi)u(t-2\pi)$$

wkt

$L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$

5) $\dfrac{s e^{-s/2} + \pi e^{-s}}{s^2+\pi^2}$

$\Rightarrow L^{-1}\left[e^{-s/2}\dfrac{s}{s^2+\pi^2}\right] + L^{-1}\left[e^{-s}\dfrac{\pi}{s^2+\pi^2}\right] \longrightarrow (1)$

wkt

$L^{-1}\left[\dfrac{s}{s^2+\pi^2}\right] = \cos \pi t, \quad L^{-1}\left[\dfrac{\pi}{s^2+\pi^2}\right] = \sin \pi t$

$(1) \Rightarrow$

$L^{-1}\left[\dfrac{s e^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right] = \cos \pi(t-1/2)\, u(t-1/2) + \sin \pi(t-1) u(t-1)$

$\cos(\pi t - \pi/2)\; u(t-1/2) + \sin(\pi t - \pi)\; u(t-1)$

$= \sin \pi t \; u(t-1/2) - \sin \pi t \; u(t-1)$

6) $L^{-1}\left[\dfrac{(1-e^{-s})(2-e^{-2s})}{s^3}\right] = t^2 - (t-1)^2 u(t-1) - \dfrac{(t-2)^2 u(t-2)}{2}$

$+ \dfrac{(t-3)^2\, u(t-3)}{2}$

(2)

Inverse by completing the square

If $L[f(t)] = f(s)$

then $L[e^{at} f(t)] = f(s-a)$

$\Rightarrow L^{-1}[f(s)] = f(t) \longrightarrow (1)$

$L^{-1}[f(s-a)] = e^{at} f(t) \longrightarrow (2)$

$L^{-1}[f(s-a)] = e^{at} L^{-1}[f(s)] \longrightarrow (3)$

$\{ \; L^{-1}[f(s+a)] = e^{-at} L^{-1}[f(s)] \} \longrightarrow (4)$

working procedure

1) Given $f(s) = \dfrac{\phi(s)}{(ps^2+qs+r)}$

2) first Express (ps^2+qs+r) in the form $(s-a)^2 \pm b^2$ & Later express $\phi(s)$ in terms of $(s-a)$ so that the given fun of 's' reduces to a function of $(s-a)$.

3) use eqn (3) or (4) to obtain the results

4) If $f(s) = \dfrac{\phi(s)}{4(s-a)}$, only need to express $\phi(s)$ in terms of $(s-a)$ to compute the inverse transform.

Examples

Find the Inverse L.T of the following functions

1) $\dfrac{s+5}{s^2-6s+13}$

$$s^2-6s+13$$
$$= s^2-6s+9-9+13$$
$$= (s-3)^2+4$$
$$= (s-3)^2+2^2$$

$$L^{-1}\left[\frac{(s+5)}{s^2-6s+13}\right] = L^{-1}\left[\frac{s+5}{(s-3)^2+2^2}\right]$$

(numerator can be express in terms of $(s-3)$

$$\therefore L^{-1}\left[\frac{(s-3)+3+5}{(s-3)^2+2^2}\right] = L^{-1}\left[\frac{(s-3)+8}{(s-3)^2+2^2}\right]$$

Here $a=3$ and $(s-3)$ changes to s

$$= e^{3t} L^{-1}\left[\frac{s+8}{s^2+2^2}\right] = e^{3t}\left[L^{-1}\left(\frac{s}{s^2+2^2}\right) + 8L^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$$

$$= e^{3t}\left[\cos 2t + 4 \sin 2t\right]$$

2)
$$\frac{s+2}{s^2-4s+13}$$

$s^2-4s+13$
$= s^2-4s+4+9$
$= (s-2)^2+3^2$

$$L^{-1}\left[\frac{s+2}{s^2-4s+13}\right] = L^{-1}\left[\frac{s+2}{(s-2)^2+3^2}\right]$$

$$= L^{-1}\left[\frac{s-2+2+2}{(s-2)^2+3^2}\right] = L^{-1}\left[\frac{(s-2)+4}{(s-2)^2+3^2}\right]$$

$$= L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]$$

$a=2$, $(s-2) \to s$

$$= e^{2t} L^{-1}\left[\frac{s}{s^2+3^2}\right] + 4e^{2t} L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

3)
$$\frac{s+1}{s^2+6s+9}$$

s^2+6s+9
$= (s+3)^2$

$$L^{-1}\left[\frac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\frac{s+1}{(s+3)^2}\right] =$$

$$= L^{-1}\left[\frac{s+3-3+1}{(s+3)^2}\right] = L^{-1}\left[\frac{s+3-2}{(s+3)^2}\right]$$

apply property $(s+3)$ changes to s & $a=-3$

$$= e^{-3t} L^{-1}\left[\frac{s-2}{s^2}\right]$$

$$= e^{-3t}\left[L^{-1}\left[\frac{s}{s^2}\right] - L^{-1}\left[\frac{2}{s^2}\right]\right]$$

$$= e^{-3t}\left[L^{-1}(1/s) - 2L^{-1}(1/s^2)\right]$$

$$= e^{-3t}[1-2t]$$

4) $\dfrac{(s+2)\,e^{-s}}{(s+1)^4}$

soln :- let $f(s) = \dfrac{s+2}{(s+1)^4}$

first we find $L^{-1}[f(s)] = f(t)$

$L^{-1}\left[\dfrac{s+2}{(s+1)^4}\right] = L^{-1}\left[\dfrac{(s+1)+1}{(s+1)^4}\right]$ ∴ Here $a=1$, $s+1 \to s$

$= e^{-t} L^{-1}\left[\dfrac{s+1}{s^4}\right]$

consider

$= e^{-t}\left\{L^{-1}\left[\dfrac{s}{s^4}\right] + L^{-1}\left[\dfrac{1}{s^4}\right]\right\}$

$= e^{-t}\left[L^{-1}\left[\dfrac{1}{s^3}\right] + L^{-1}\left[\dfrac{1}{s^4}\right]\right]$ $\left\{ L^{-1}\left(\dfrac{1}{s^{n+1}}\right) = \dfrac{t^n}{n!} \right\}$

$= e^{-t}\left[\dfrac{t^2}{2!} + \dfrac{t^3}{3!}\right] = e^{-t}\left[\dfrac{t^2}{2} + \dfrac{t^3}{6}\right]$

wkt $L^{-1}[e^{-s} f(s)] = f(t-1)\,u(t-1)$

$\therefore L^{-1}\left[e^{-s}\dfrac{s+2}{(s+1)^4}\right] = e^{-(t-1)}\left[\dfrac{(t-1)^2}{2} + \dfrac{(t-1)^3}{6}\right] u(t-1)$

5) $\dfrac{2s+1}{s^2+3s+1}$

consider $s^2+3s+1 = s^2+3s+\dfrac{9}{4} - \dfrac{9}{4} + 1$

$= (s+3/2)^2 - \dfrac{5}{4}$

$-\dfrac{9}{4}+1$
$\dfrac{-9+4}{4}$
$\dfrac{-5}{4}$

$$L^{-1}\left[\frac{2s+1}{s^2+3s+1}\right] = L^{-1}\left[\frac{2s+1}{\left(s+3/2\right)^2-\left(\sqrt{5}/2\right)^2}\right]$$

$$= L^{-1}\left[\frac{2(s+3/2)-2}{\left(s+3/2\right)^2-\left(\sqrt{5}/2\right)^2}\right]$$

$2s+1$
$= 2(s+3/2)-2$
$= 2s+3-2$
$= 2s+1$

$$= e^{-3t/2}\left[L^{-1}\left[\frac{2s-2}{(s)^2-\left(\sqrt{5}/2\right)^2}\right]\right\}$$

$$= e^{-3/2t}\left\{L^{-1}\left[\frac{2s}{s^2-\left(\sqrt{5}/2\right)^2}\right] - 2L^{-1}\left[\frac{1}{s^2-\left(\sqrt{5}/2\right)^2}\right]\right\}$$

$$= e^{-3/2t}\left[2L^{-1}\left[\frac{s}{s^2-\left(\sqrt{5}/2\right)^2}\right] - 2L^{-1}\left[\frac{1}{s^2-\left(\sqrt{5}/2\right)^2}\right]\right\}$$

$$= e^{-3/2t}\left[2\cosh\left(\frac{\sqrt{5}}{2}t\right) - \frac{2}{\sqrt{5}/2}\sinh\left(\frac{\sqrt{5}t}{2}\right)\right]$$

$$= e^{-\frac{3}{2}t}\left[2\cosh\left(\frac{\sqrt{5}}{2}t\right) - \frac{4}{\sqrt{5}}\sinh\left(\frac{\sqrt{5}t}{2}\right)\right]$$

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6) $\dfrac{7s+4}{4s^2+4s+9}$

$(s+1/2)^2+2$
$s^2+s+1/4+2$
$= \frac{1+2}{4}$

consider, $4s^2+4s+9 = 4(s^2+s+9/4) = 4\left(\underbrace{s^2+s+\frac{1}{4}}-\frac{1}{4}+\frac{9}{4}\right]$

$$= 4\left[(s+1/2)^2+\cancel{8}^{2}/\cancel{4}\right]$$

$$= 4\left((s+1/2)^2+2\right)$$

& also

$$7s+4 = 7(s+1/2)+1/2.$$

$$\therefore L^{-1}\left[\frac{7s+4}{4s^2+4s+9}\right] = \frac{1}{4}L^{-1}\left[\frac{7(s+1/2)+1/2}{(s+1/2)^2+2}\right] \quad a=-1/2$$

$$= \frac{e^{-t/2}}{4} L^{-1}\left[\frac{7s+1/2}{s^2+(\sqrt{2})^2}\right]$$

$$= \frac{e^{-t/2}}{4}\left\{7L^{-1}\left(\frac{s}{s^2+(\sqrt{2})^2}\right)+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]\right\}$$

$$= \frac{e^{-t/2}}{4}\left[7\cos(\sqrt{2}t)+\frac{1}{2\sqrt{2}}\sin(\sqrt{2}t)\right]$$

Inverse by the method of partial fractions

The method of partial fractions is a technique of converting an algebraic function $\frac{\phi(s)}{\psi(s)}$ [where degree of $\phi(s)$ is less than that of $\psi(s)$] into a sum. Depending on the nature of terms in $\psi(s)$ we have to split into a sum of various terms with constants A, B, C, D - - - which can be determined. Later the inverse is found term by term.

Find the inverse L.T of the following function

i) $\dfrac{1}{s(s+1)(s+2)(s+3)}$

Let $\dfrac{1}{s(s+1)(s+2)(s+3)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2} + \dfrac{D}{s+3}$

multiplying by $s(s+1)(s+2)(s+3)$, we get

$1 = A(s+1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s+1)(s+3) + Ds(s+1)(s+2)$

put $s=0$

$1 = A(1)(2)(3) + B(0) + C(0) + D(0) \Rightarrow A = 1/6$

put $s=-1$

$1 = A(0) + B(-1)(-1+2)(-1+3) + C(0) + D(0)$

$\Rightarrow B = -1/2$

put $s=-2 \Rightarrow 1 = C(2) \Rightarrow C = 1/2$

put $s=-3 \Rightarrow 1 = D(-6) \Rightarrow D = -1/6$

(1) $\Rightarrow$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = L^{-1}\left[\frac{1/6}{s}\right] + L^{-1}\left[\frac{-1/2}{s+1}\right] + L^{-1}\left[\frac{1/2}{s+2}\right] + L^{-1}\left[\frac{-1/6}{s+3}\right]$$

$$= \frac{1}{6}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{6}(1) - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t}$$

2) $\dfrac{s^2}{(s^2+1)(s^2+4)}$

Soln: Let $s^2 = t$ [for convenience]

$$\frac{t}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4} \longrightarrow (1)$$

x'y $(t+1)(t+4)$ on both side

$$\Rightarrow \quad t = A(t+4) + B(t+1)$$

put $t=-1 \Rightarrow -1 = A(-1+4) + B(0)$

$$\Rightarrow \boxed{A = -1/3}$$

put $t = -4 \Rightarrow -4 = A(0) + B(-4+1) \Rightarrow \boxed{B = 4/3}$

(1) $\Rightarrow \dfrac{t}{(t+1)(t+4)} = \dfrac{-1}{3}\dfrac{1}{t+1} + \dfrac{4}{3}\cdot\dfrac{1}{t+4}$

substitute $t = s^2$ & taking Inverse L.T

$$L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] = \frac{-1}{3}\cdot L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{4}{3} L^{-1}\left[\frac{1}{s^2+4_{2^2}}\right]$$

$$= \frac{-1}{3}\sin t + \frac{\overset{2}{\cancel{4}}}{3}\frac{1}{\cdot\cancel{2}}\sin 2t$$

$$= \frac{1}{3}\left[2\sin 2t - \sin t\right]$$

8) $\frac{4s+5}{(s+1)^2(s+2)}$

Let $\frac{4s+5}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s+2} \rightarrow (1)$

x'y by $(s+1)^2(s+2)$,

$4s+5 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \rightarrow (*)$

put $s=-1 \Rightarrow 1 = B(1) \Rightarrow \boxed{B=1}$

put $s=-2 \Rightarrow -3 = C(1) \Rightarrow \boxed{C=-3}$

To find A)

Equating the co-efficient of s^2 on both sides of (*).

$0 = A + C \Rightarrow 0 = A - 3 \Rightarrow \boxed{A=3}$

substitute all these values in (1) & taking Inverse L.T.

$$L^{-1}\left[\frac{4s+5}{(s+1)^2(s+2)}\right] = L^{-1}\left[\frac{3}{(s+1)} + \frac{1}{(s+1)^2} + \frac{-3}{s+2}\right]$$

$$= 3L^{-1}\left[\frac{1}{(s+1)}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] - 3L^{-1}\left[\frac{1}{s+2}\right] \rightarrow (2)$$

where $L^{-1}[f(s+a)] = e^{-at} L^{-1}[f(s)]$

$L^{-1}[f(s+1)] = e^{-t} L^{-1}(1/s^2] = e^{-t}.t$

from (2)

$$L^{-1}\left[\frac{4s+5}{(s+1)^2(s+2)}\right] = 3e^{-t} + e^{-t}.t - 3e^{-2t}$$

(3) > $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$

consider $s^3+s^2-2s = s(s^2+s-2) = s(s-1)(s+2)$

s^2+s-2
$s^2+2s-s-2$
$s(s+2)-1(s+2)$
$(s-1)(s+2)$

$$\frac{2s^2+5s-4}{s^3+s^2-2s} = \frac{2s^2+5s-4}{s(s-1)(s+2)}$$

$$\frac{2s^2+5s-4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s+2)}$$

x'y by $s(s-1)(s+2)$

$2s^2+5s-4 = A(s-1)(s+2) + Bs(s+2) + Cs(s-1)$

put $s=0 \Rightarrow -4 = A(-2) \Rightarrow A=2$

put $s=1 \Rightarrow 3 = B(3) \Rightarrow B=1$

put $s=-2 \Rightarrow -6 = C(6) \Rightarrow C=-1$

$$L^{-1}\left[\frac{2s^2+5s-4}{s(s-1)(s+2)}\right] = L^{-1}\left[\frac{2}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{-1}{s+2}\right]$$

$$= 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 2 + e^{t} - e^{-2t}$$

5> $\dfrac{s+2}{s^2(s+3)}$

consider $\dfrac{s+2}{s^2(s+3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3} \rightarrow (1)$

x'y by $s^2(s+3)$

$\Rightarrow (s+2) = As(s+3) + B(s+3) + Cs^2 \rightarrow (*)$

put $s=0 \Rightarrow 2 = B(3) \Rightarrow B = 2/3$

put $s=-3 \Rightarrow -1 = C(9) \Rightarrow C = \ldots$

Equating the co-efficient of s^2 on b.s of (*).

$0 = A + C \Rightarrow A = -C = 1/9$

substitute in (1) & taking inverse LT

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9}L^{-1}\left[\frac{1}{s}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{9}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{9}(1) + \frac{2}{3}.6t - \frac{1}{9}e^{-3t}$$

6) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$.

$$\Rightarrow \frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+2s+5} \rightarrow (1)$$

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1) \rightarrow (*)$$

put $s=1 \Rightarrow 8 = A(8) \Rightarrow A = 1$

put $s=0 \Rightarrow 3 = 5A - C \Rightarrow C = 2.$

Equating the co-efficient of s^2 on both sides (*),

$0 = A + B \Rightarrow B = -1.$

$\therefore$ (1) $\Rightarrow$

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{-s+2}{(s^2+2s+5)}$$

$$= \frac{1}{(s-1)} + \frac{2-s}{(s+1)^2+4}$$

$$= \frac{1}{(s-1)} + \frac{3-(s+1)}{(s+1)^2+2^2}$$

s^2+2s+5

$= s^2+2s+1+4$

$= (s+1)^2 + 2^2$

$2-s$

$= 3-(s+1)$

Taking Inverse L.T

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{3-(s+1)}{(s+1)^2+4}\right]$$

$$= e^t + e^{-t} L^{-1}\left[\frac{3-s}{s^2+4}\right]$$

$$= e^t + e^{-t}\left[3L^{-1}\left(\frac{1}{s^2+4}\right) - L^{-1}\left(\frac{s}{s^2+4}\right)\right]$$

$$= e^t + e^{-t}\left[\frac{3}{2}\sin 2t - \cos 2t\right]$$

7) $L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right] = \frac{1}{3}\left[e^{-t}\sin t\right] + \frac{1}{3}e^{-t}\sin 2t$

(put $t = s^2+2s$)

8) $L^{-1}\left[\frac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}\right] = \left[2e^{t-3} - 2\cos(t-3) + \sin(t-3)\right]u(t-3)$

Computation of the inverse transform by using convolution theorem :-

working procedure

1) Given function is expressed as the product of two functions say $f(s)$ & $g(s)$.

2) Find $L^{-1}[f(s)] = f(t)$ & $L^{-1}[g(s)] = g(t)$

3) Apply the convolution theorem in one of the form

$$L^{-1}[f(s) \cdot g(s)] = \int_{u=0}^{t} f(u) \cdot g(t-u)\,du$$

4) Evaluate the convolution ~~theorem~~ integral to obtain the required inverse.

Examples

Using convolution theorem obtain the inverse L.T of the following functions

i) $\dfrac{1}{s(s^2+a^2)}$

soln: Let $f(s) = \frac{1}{s}$ and $g(s) = \dfrac{1}{s^2+a^2}$

Taking inverse L.T

$$L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s}\right] = 1 = f(t) \quad \& \quad L^{-1}[g(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a} = g(t)$$

we have convolution theorem,

$$L^{-1}[f(s) \cdot g(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\,du = \int_{u=0}^{t} 1\, \frac{\sin a(t-u)}{a}\,du$$

$$= \int_{u=0}^{t} \frac{\sin(at-au)}{a}\,du$$

$$= \frac{1}{a} \cdot \left[\frac{\cos(at-au)}{a}\right]^{t}$$

$$= \frac{1}{a^2}\left[\cos(\cancel{a}t - at) - \cos(at - 0)\right]$$

$$= \frac{1}{a^2}\left[\cos(0) - \cos at\right]$$

$$L^{-1}[f(s).g(s)] = \frac{1}{a^2}[1 - \cos at]$$

2) $\dfrac{s}{(s^2+a^2)^2}$

Soln: Let $f(s) = \dfrac{1}{s^2+a^2}$ & $g(s) = \dfrac{s}{(s^2+a^2)}$

$\Rightarrow L^{-1}[f(s)] = f(t) = L^{-1}\left[\dfrac{1}{s^2+a^2}\right]$ & $L^{-1}(g(s)) = L^{-1}\left[\dfrac{s}{s^2+a^2}\right]$

$f(t) = \dfrac{\sin at}{a}$ $\qquad g(t) = \cos at$

w.k.t, by convolution theorem.

$$L^{-1}[f(s).g(s)] = \int_{u=0}^{t} f(u).g(t-u)\,du.$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a}.\cos(at - au)\,du$$

$$= \frac{1}{2a}\int_{u=0}^{t}[\sin(au + at - au) + \sin(au - at + au)]\,du$$

$\frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$= \frac{1}{2a}\int_{u=0}^{t}[\sin at + \sin(2au - at)]\,du.$$

$$= \frac{1}{2a}\left\{\sin at.[u]_{u=0}^{t} - \left[\frac{\cos(2au - at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2a}\left\{\sin at.(t-0) - \frac{1}{2a}[\cos at - \cos at]\right\}$$

$$= \frac{t.\sin at}{2a}$$

3) $\dfrac{1}{(s-1)(s^2+1)}$

Soln. Let $f(s) = \dfrac{1}{s-1}$, $g(s) = \dfrac{1}{s^2+1}$

$\Rightarrow$ Taking Inverse L.T

$L^{-1}[f(s)] = L^{-1}\left[\dfrac{1}{s-1}\right] = e^t = f(t)$ & , $L^{-1}[g(s)] = L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t$
$= g(t)$

by applying convolution theorem, we have

$$L^{-1}[f(s) \cdot g(s)] = \int_{u=0}^{t} f(u) \cdot g(t-u) \, du$$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_{u=0}^{t} e^u \cdot \sin(t-u) \, du \longrightarrow (1)$$

wkt

$$\int e^{ax} \sin(bx+c) \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin(bx+c) - b\cos(bx+c)\right]$$

(1) $\Rightarrow$ $a = 1$, $b = -1$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \left[\frac{e^u}{(1)^2+(-1)^2}\left[(1)\sin(t-u) - (-1)\cos(t-u)\right]\right]_0^t$$

$$= \left[\frac{e^u}{2}\left[\sin(t-u) + \cos(t-u)\right]\right]_0^t$$

$$= \frac{1}{2}\left\{e^t\left[\sin(t-t) + \cos(t-t)\right] - e^0\left[\sin(t-0) + \cos(t-0)\right]\right\}$$

$$= \frac{1}{2}\left[e^t(0+1) - 1(\sin t + \cos t)\right]$$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{1}{2}\left[e^t - \sin t - \cos t\right]$$

(4) $\dfrac{s+2}{(s^2+4s+5)^2}$

Let $f(s) = \dfrac{s+2}{s^2+4s+5}$ & $g(s) = \dfrac{1}{s^2+4s+5}$ $\Big\} \rightarrow (1)$

Consider s^2+4s+5 = ~~s^2+4s+~~

$a=1, b=4, c=5$, $\Rightarrow b^2-4ac = 16-4(1)(5) = 16-20 = -4 = \underline{-ve}$

$$s^2+4s+5 = s^2+4s+5-4+4 \quad \text{[add \& subtract 4]} \quad \left(\frac{\text{co-eff of } s}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 2^2 = 4$$

$$= s^2+4s+4+1$$

$$= (s+2)^2+1$$

$\therefore f(s) =$ s

$(1) \Rightarrow$

$L^{-1}[f(s)] = L^{-1}\left[\dfrac{s+2}{(s+2)^2+1}\right]$ & $L^{-1}[g(s)] = L^{-1}\left[\dfrac{1}{(s+2)^2+1}\right]$

$f(t) = e^{-2t} \cdot L^{-1}\left[\dfrac{s}{s^2+1}\right]$

$g(t) = L^{-1}\left[\dfrac{1}{(s+2)^2+1}\right]$

$f(t) = e^{-2t} \cos t$

$g(t) = e^{-2t} L^{-1}\left[\dfrac{1}{s^2+1}\right]$

$g(t) = e^{-2t} \sin t$

Now by applying convolution theorem

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \int_{u=0}^{t} e^{-2u} \cos u \, e^{-2(t-u)} \sin(t-u) \, du$$

$$= e^{-2t} \int_{u=0}^{t} \sin(t-u) \cdot \cos u \cdot e^{-2u} e^{2u} \, du$$

$$= e^{-2t}\int_{u=0}^{t} \sin(t-u)\cos u \, du$$

$$= \frac{e^{-2t}}{2}\int_{u=0}^{t} \left[\sin(t-u+u) + \sin(t-u-u)\right] du$$

$$\left[\sin A \cdot \cos B = \tfrac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)\right]$$

$$= \frac{e^{-2t}}{2}\int_{u=0}^{t} \left[\sin t + \sin(t-2u)\right] du$$

$$= \frac{e^{-2t}}{2}\left[\sin t \,[u]_{u=0}^{t} + \left[\frac{\cos(t-2u)}{2}\right]_{u=0}^{t} \right]$$

$$= \frac{e^{-2t}}{2}\left[(t \sin t - 0 \cdot \sin t) + \frac{1}{2}\left[\cos(t-2t) - \cos(t-0)\right] \right]$$

$$= \frac{e^{-2t}}{2}\left[t \sin t + \frac{1}{2}\left[\cos t - \cos t\right] \right]$$

$$= \frac{e^{-2t}\, t \sin t}{2}$$

Laplace transform of the derivative

A differential equation with a initial conditions is called an initial value problem

working procedure

1) The given differential Eqn is expressed in the notation $y'(t)$, $y''(t)$ $y'''(t)$, for the derivatives

2) Take L.T on both side of the given eqn

3) Use the expression for $L[y'(t)]$ $L[y''(t)]$ - -

where $L[y'(t)] = s L[y(t)] - y(0)$

$$L[y''(t)] = s^2 L[y(t)] - s y(0) - y'(0)$$

$$L[y'''(t)] = s^3 L[y(t)] - s^2 y(0) - s y'(0) - y''(0)$$

4) substitute the given initial conditions & simplify to obtain $L[y(t)]$ as a function of s.

5) find the inverse to obtain $y(t)$.

Examples

1) Solve by using Laplace transforms

$$\frac{d^2y}{dt^2} + k^2 y = 0 \text{ given that } y(0) = 2, \ y'(0) = 0$$

soln: The given eqn is $y''(t) + k^2 y(t) = 0$

Take L.T on b.s

$$L[y''(t)] + k^2 L[y(t)] = L[0]$$

$$[s^2 L[y(t)] - s[y(0)] - y'(0)] + k^2 L[y(t)] = 0$$

use the given initial condn. then above eqn Becomes.

$$\left[s^2 L[y(t)] - s(2) - 0\right] + k^2 L[y(t)] = 0$$

$$(s^2+k^2) L[y(t)] - 2s = 0$$

$$L[y(t)] = \frac{2s}{s^2+k^2} \Rightarrow y(t) = L^{-1}\left[\frac{2s}{s^2+k^2}\right]$$

$$y(t) = 2 L^{-1}\left[\frac{s}{s^2+k^2}\right]$$

$$\Rightarrow y(t) = \underline{2\cos kt}.$$

2) solve $y''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$ by using L.T method.

Soln: The given eqn is

$$y'''(t) + 2y''(t) - y'(t) - 2y(t) = 0$$

Taking L.T.

$$L[y'''(t)] + 2L[y''(t)] - L[y'(t)] - 2L[y(t)] = L[0]$$

$$\Rightarrow \left[s^3 L[y(t)] - s^2 y(0) - s y'(0) - y''(0)\right]$$

$$+ 2\left[s^2 L[y(t)] - s y(0) - y'(0)\right] - \left[s L(y(t)) - y(0)\right]$$

$$- 2L[y(t)] = 0$$

Now using the given initial condn

$$\left[s^3 L[y(t)] - \cancel{s^2 y(0)} - s y'(0)^{\to 0} - 6\right]$$

$$+ 2\left[s^2 L[y(t)] - \cancel{s y(0)} - 0\right] - \left[s L[y(t)] - 0\right] - 2L[y(t)] = 0$$

$$\Rightarrow (s^3 + 2s^2 - s - 2)\, L[y(t)] - 6 = 0$$

$$L[y(t)]\left[s^2(s+2) - 1(s+2)\right] = 6$$

$$L[y(t)]\left[(s+2)(s^2-1)\right] = 6$$

$$L[y(t)]\{(s+2)(s-1)(s+1)\} = 6 \qquad (a^2 - b^2 = (a+b)(a-b))$$

$$\therefore L[y(t)] = \frac{6}{(s+2)(s-1)(s+1)}$$

$$\Rightarrow y(t) = L^{-1}\left[\frac{6}{(s+2)(s-1)(s+1)}\right] \to (1)$$

consider

$$\frac{6}{(s+2)(s-1)(s+1)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{s+1}$$

x'y by $(s+2)(s-1)(s+1)$

$$\Rightarrow 6 = A(s-1)(s+1) + B(s+2)(s+1) + C(s+2)(s-1)$$

put $s = -2 \Rightarrow 6 = A(-3)(-1) \Rightarrow A = 2$

put $s = 1 \Rightarrow 6 = B(3)(2) \Rightarrow B = 1$

put $s = -1 \Rightarrow 6 = C(1)(-2) \Rightarrow C = -3$

$$\therefore \frac{6}{(s+2)(s-1)(s+1)} = \frac{2}{s+2} + \frac{1}{s-1} + \frac{-3}{s+1}$$

apply Inverse L.T

$$L^{-1}\left[\frac{6}{(s+2)(s-1)(s+1)}\right] = 2L^{-1}\left[\frac{1}{(s+2)}\right] + L^{-1}\left[\frac{1}{s-1}\right]$$

$$-3 \, L^{-1}\left[\frac{1}{s+1}\right]$$

$$\Rightarrow \quad y(t) = 2e^{-2t} + e^{t} - 3e^{-t}$$

3) Solve the following IVP by using L.T

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}, \quad y(0)=0, \quad y'(0)=0$$

Soln: Given, $y''(t) + 4\, y'(t) + 4y(t) = e^{-t}$

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L[e^{-t}]$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 4[sL[y(t)] - y(0)] + 4L[y(t)] = \frac{1}{s+1}$$

$$s^2L[y(t)] - 0 - 0 + 4sL[y(t)] - 4(0) + 4L[y(t)] = \frac{1}{s+1}$$

$$L[y(t)]\,[s^2+4s+4] = \frac{1}{s+1}$$

$$L[y(t)] = \frac{1}{(s+1)(s^2+4s+4)}$$

s^2+4s+4
$= (s+2)^2$

$$y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] \leftarrow ①$$

(A1)

Let
$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \quad \text{---} (2)$$

x'y by $(s+1)(s+2)^2$

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Put $s = -1 \Rightarrow A = 1$

$s = -2 \Rightarrow C = -1$

Put $s = 0 \Rightarrow 1 = A(4) + B(2) + C(1)$

$1 = 1(4) + B(2) - 1(1)$

$\Rightarrow B = -1$

(2) $\Rightarrow$

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + \frac{-1}{s+2} + \frac{-1}{(s+2)^2}$$

(1) $\Rightarrow$

$$y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$\Rightarrow y(t) = e^{-t} - e^{-2t} - e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-t} - e^{-2t} - e^{-2t}(t)$$

$$y(t) = e^{-t} - (1+t)e^{-2t}$$

solve the foll IVP by using L.T

4) $x'' - 2x' + x = e^{2t}$ with $x(0) = 0$, $x'(0) = -1$

Ans: $x(t) = -e^{t} - 2e^{t}\,t + e^{2t}$

5) $y'' + 4y' + 3y = e^{-t}$, $y(0) = 1 = y'(0)$

Ans: $y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}\,t - \frac{3}{4}e^{-3t}$

6) $y'' + 5y' + 6y = 5e^{2x}$, $y(0) = 2$, $y'(0) = 1$

Ans: $y(x) = \frac{1}{4}e^{2x} + \frac{23}{4}e^{-2x} - 4e^{-3x}$

Inverse Laplace transforms

Defn :- If $L[f(t)] = f(s)$ then $f(t)$ is called the inverse Laplace transform of $f(s)$ and is denoted by $L^{-1}[f(s)]$.

Thus,

$$L[f(t)] = f(s) \iff L^{-1}[f(s)] = f(t)$$

Ex :-

1) $L[1] = 1/s \Rightarrow L^{-1}[1/s] = 1$

2) $L[\cos at] = \dfrac{s}{s^2+a^2} \Rightarrow L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at$.

Basic table of Inverse Laplace transforms :-

	Function	Inverse transform
1)	$1/s$	1
2)	$\dfrac{1}{s-a}$	e^{at}
3)	$\dfrac{1}{s+a}$	e^{-at}
4)	$\dfrac{s}{s^2+a^2}$	$\cos at$
5)	$\dfrac{s}{s^2-a^2}$	$\cosh at$
6)	$\dfrac{1}{s^2+a^2}$	$\dfrac{\sin at}{a}$
7)	$\dfrac{1}{s^2-a^2}$	$\dfrac{\sinh at}{a}$
8)	$\dfrac{1}{s^{n+1}}$, $n=1,2,3\ldots$	$\dfrac{t^n}{n!}$

Examples

1) $L^{-1}\left[\frac{1}{s-1}\right] = e^{t}$

2) $L^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t$

3) $L^{-1}\left[\frac{1}{s^2+5}\right] = \frac{1}{\sqrt{5}} \sin(\sqrt{5}\,t)$

4) $L^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$

5) $L^{-1}\left[\frac{s}{s^2-16}\right] = \cosh$

6) $L^{-1}\left[\frac{1}{s^2-36}\right] = \frac{1}{6}\sinh 6t$

7) $L^{-1}\left[\frac{1}{s^4}\right] = \frac{t^3}{3!}$

Note :- Property

$$L^{-1}\left[c_1 f(s) + c_2 g(s)\right] = c_1 L^{-1}[f(s)] + c_2 L^{-1}[g(s)]$$

Find the Inverse Laplace transform of the following functions

1) $\frac{1}{s+2} + \frac{3}{2s+5} - \frac{4}{3s-2}$

soln Taking inverse L.T

$$L^{-1}\left[\frac{1}{s+2}\right] + L^{-1}\left[\frac{3}{2(s+5/2)}\right] - L^{-1}\left[\frac{4}{3(s-2/3)}\right]$$

$$= e^{-2t} + \frac{3}{2} L^{-1}\left[\frac{1}{s+5/2}\right] - \frac{4}{3} L^{-1}\left[\frac{1}{s-2/3}\right]$$

$$= e^{-2t} + \frac{3}{2} e^{-5/2\, t} - \frac{4}{3} e^{2/3\, t}$$

2) $\frac{s+2}{s^2+36} + \frac{4s-1}{s^2+25}$

soln:

$$= \frac{s}{s^2+36} + \frac{2}{s^2+36} + \frac{4s}{s^2+25} - \frac{1}{s^2+25}$$

Taking Inverse L.T

$$\Rightarrow L^{-1}\left[\frac{s}{s^2+6^2}\right] + 2L^{-1}\left[\frac{1}{s^2+6^2}\right] + 4L^{-1}\left[\frac{s}{s^2+5^2}\right] - L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$= \cos 6t + \frac{2}{6}\cdot \sin 6t + 4\cos 5t - \frac{1}{5}\sin 5t$$

$$= \cos 6t + \underline{\tfrac{1}{3}\sin 6t + 4}\cos 5t - \tfrac{1}{5}\sin 5t$$