Module - 4

Laplace Transforms

Definition :- If $f(t)$ is a real valued function defined for all $t \geq 0$ then the Laplace transform of $f(t)$ denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} . f(t) . dt .$$

provided that the integral exists. 's' is a parameter which may be a real or complex number.

Background of L.T

L.T in mathematics, a particular integral transform invented by the French mathematician Pierre-Simon Laplace [1749-1827] and systematically developed by the British physicist Oliver Heaviside [1850-1925], to simplify the solution of many differential equations that describe physical process.

Applications :-

L.T methods have a key role to play in the modern approach to the analysis and design of Engineering system.

Some applications of L.T in science & Engineering fields

1) Analysis of electronic circuits
2) system modelling
3) Digital processing
4) control Engg
5) communication Engg.
6) Nuclear physics.

$\sin(-4x) = -\sin 4x$

2)

Laplace transform of some standard functions

1) $L[a] = \frac{a}{s}$

2) $L[1] = \frac{1}{s}$

3) $L[e^{at}] = \frac{1}{s-a}$ where $s > a$

4) $L[e^{-at}] = \frac{1}{s+a}$

5) $L[\sin at] = \frac{a}{s^2+a^2}$ where $s>0$

6) $L[\cos at] = \frac{s}{s^2+a^2}$ where $s>0$

7) $L[\sinh at] = \frac{a}{s^2-a^2}$

8) $L[\cosh at] = \frac{s}{s^2-a^2}$

9) $L[t^n] = \frac{n!}{s^{n+1}}$ where n is +ve integer

10) $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ where 'n' is constant

9) $L[t^n] = \frac{n!}{s^{n+1}}$

$\int_0^\infty e^{-st} f(t)\,dt$

$\int_0^\infty e^{-st} a\,dt$

$\left[\frac{ae^{-st}}{-s}\right]_0^\infty$

$[0 + \frac{1}{s}(a)]$

$L(a) = \frac{a}{s}$

Examples

1) $L[4] = \frac{4}{s}$

2) $L[e^{-t}] = \frac{1}{s+1}$

3) $L[2\cosh 2t] = \frac{2s}{s^2-4}$

4) $L[e^{2t}] = \frac{1}{s-2}$

5) $L[3e^{4t}] = \frac{3}{s-4}$

6) $L[3\sinh 2t] = \frac{6}{s^2-4}$

7) $L[\cos t] = \frac{s}{s^2+1}$

8) $L[t^5] = \frac{5!}{s^6} = \frac{120}{s^6}$

9) $L[e^{-t}\cosh 3t] = L\left(e^{-t}\left[\frac{e^{3t}+e^{-3t}}{2}\right]\right)$

$= \frac{1}{2}\left[L[e^{2t}] + L[e^{-4t}]\right]$

$= \frac{1}{2}\left[\frac{1}{s-2} + \frac{1}{s+4}\right]$

10) $L[e^{2t}\sinh 4t]$

$L\left[e^{2t}\left(\frac{e^{4t}-e^{-4t}}{2}\right)\right]$

$= \frac{1}{2}\left[L(e^{6t}) - L(e^{-2t})\right]$

$= \frac{1}{2}\left[\frac{1}{s-6} - \frac{1}{s+2}\right]$

10) $L[e^{-2t}\sinh 4t]$

$= L\left[e^{-2t}\left(\frac{e^{4t}-e^{-4t}}{2}\right)\right]$

$= \frac{1}{2}\left[L(e^{2t}) - L[e^{-6t}]\right]$

$= \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+6}\right]$

* Find the Laplace transform of the following functions:-

1) $\cosh^2 3t$

Let $f(t) = \cosh^2 3t = (\cosh 3t)^2$

$\left[\text{wkt } \cosh(at) = \frac{e^{at} + e^{-at}}{2}\right]$

$\therefore f(t) = \left(\frac{e^{3t} + e^{-3t}}{2}\right)^2 = \frac{1}{4}\left[e^{6t} + e^{-6t} + 2\, e^{6t}.e^{-6t}\right]$

$f(t) = \frac{1}{4}\left[e^{6t} + e^{-6t} + 2\right]$ 2L(1)

Apply L.T on both side.

$L[f(t)] = \frac{1}{4}\left[L[e^{6t}] + L[e^{-6t}] + L[2]\right]$

$L[\cosh^2 3t] = \frac{1}{4}\left[\frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right]$

2) $\sin 5t . \cos 2t$.

Let $f(t) = \sin 5t . \cos 2t$

wkt $\sin A . \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\therefore f(t) = \frac{1}{2}\left[\sin(5t+2t) + \sin(5t-2t)\right]$

$f(t) = \frac{1}{2}\left[\sin 7t + \sin 3t\right]$

Apply L.T.

$L[f(t)] = \frac{1}{2}\left[L[\sin 7t] + L[\sin 3t]\right]$

$= \frac{1}{2}\left\{\left[\frac{7}{s^2+49}\right] + \frac{3}{s^2+9}\right\}$

3) $\cos t.\cos 2t \cos 3t$

Let $f(t) = \cos t.\cos 2t.\cos 3t$

$\Rightarrow$ wkt $\cos A.\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

Consider. $\cos t \cos 2t = \frac{1}{2}[\cos(t+2t) + \cos(t-2t)]$

$= \frac{1}{2}[\cos(3t) + \cos(-t)]$ $(\because \cos(-\theta) = \cos\theta)$

$\cos t.\cos 2t = \frac{1}{2}[\cos 3t + \cos t]$

Again consider.

$(\cos t.\cos 2t)\cos 3t = \frac{1}{2}[\cos 3t.\cos 3t + \cos t.\cos 3t]$

$= \frac{1}{2}[\frac{1}{2}(\cos(6t) + \cos(0)) + \frac{1}{2}[\cos 4t + \cos 2t]]$

$= \frac{1}{4}[\cos 6t + 1 + \cos 4t + \cos 2t]$

apply L.T

$L[\cos t.\cos 2t.\cos 3t] = \frac{1}{4}[L[\cos 6t] + L[1] + L[\cos 4t] + L[\cos 2t]]$

$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{1}{s} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$

4) $\sin^2(2t+1)$

Let $f(t) = \sin^2(2t+1)$

wkt $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\therefore f(t) = \frac{1}{2}[1 - \cos 2(2t+1)]$

$= \frac{1}{2}[1 - \cos(4t+2)]$

$f(t) = \frac{1}{2}\left[1 - (\cos 4t . \cos 2 - \sin 4t . \sin 2)\right]$

Here we use formula

$\cos(A+B) = \cos A . \cos B - \sin A . \sin B$

$\therefore f(t) = \frac{1}{2}\left[1 - \cos 2 . \cos 4t + \sin 4t . \sin 2\right]$

Apply L.T

$L[f(t)] = \frac{1}{2}\left[L[1] - \cos 2 . L[\cos 4t] + \sin 2 . L[\sin 4t]\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \cos 2 . \frac{s}{s^2+16} + \sin 2 . \frac{4}{s^2+16}\right]$

Here $\cos 2$ & $\sin 2$ are constants

5) $(3t+4)^3 + 5^t$

Let $f(t) = (3t+4)^3 + 5^t$

wkt $(a+b)^3 = a^3 + b^3 + 3a^2 b + 3ab^2$ & $a^x = e^{\log a . x}$

$\therefore f(t) = (3t)^3 + (4)^3 + 3(3t)^2(4) + 3(3t)(4)^2 + e^{\log 5 . t}$

$f(t) = 27t^3 + 64 + 108t^2 + 144t + e^{(\log 5) . t}$

apply L.T & we use $L[t^n] = \frac{n!}{s^{n+1}}$

$L[f(t)] = 27 L[t^3] + 64 L[1] + 108 L[t^2] + 144 L[t] + L[e^{\log 5 . t}]$

$= 27 . \frac{3!}{s^{3+1}} + 64 . \frac{1}{s} + 108 . \frac{2!}{s^{2+1}} + 144 . \frac{1!}{s^{1+1}} + \frac{1}{s - \log 5}$

$= \frac{162}{s^4} + \frac{64}{s} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{1}{s - \log 5}$

<u>Properties of Laplace transforms.</u>

If $L[f(t)] = \bar{f}(s)$ then

1) $L[e^{at} f(t)] = \bar{f}(s-a)$

$\|^{ly}$ $L[e^{-at} f(t)] = \bar{f}(s+a)$

This property is known as the <u>shifting property.</u>

$e^{2t} \sin t$

$L\{\sin t\} = \frac{1}{s^2+1} = \bar{f}(s)$

$e^{2t} \sin t = \frac{1}{(s-2)^2+1} = \bar{f}(s-2)$

2) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$, where n is positive integer.

This property is called the <u>derivative of the transform property.</u>

3) $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds.$

4) $L\left[\int_0^t f(t)\,dt\right] = \frac{\bar{f}(s)}{s}$

<u>Find the L.T of the following functions</u> :-

1) $e^{-3t}[2\cos 5t - 3\sin 5t]$

Let $f(t) = 2\cos 5t - 3\sin 5t$

$L[f(t)] = 2L[\cos 5t] - 3L[\sin 5t]$

$= 2\left(\frac{s}{s^2+5^2}\right) - 3\frac{5}{s^2+5^2}$

$L[f(t)] = \frac{2s-15}{s^2+25}$

By shifting property.

$L[e^{-3t} f(t)] = \left[\frac{2s-15}{s^2+25}\right]_{s \to s+3}$ by prop ①

(ie Replace s by $s+3$)

$\therefore L[e^{-3t} f(t)] = \dfrac{2(s+3)-15}{(s+3)^2+25}$ ④

$= \dfrac{2s+6-15}{(s+3)^2+25} = \dfrac{2s-9}{s^2+9+6s+25}$

$\therefore L[e^{-3t}(2\cos 5t - 3\sin 5t)] = \dfrac{2s-9}{s^2+6s+34}$

2) $e^{2t}\cos^2 t$

let $f(t) = \cos^2 t = \dfrac{1+\cos(2t)}{2}$ $\qquad$ i.e. $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$L[f(t)] = \frac{1}{2}\{L[1] + L[\cos 2t]\}$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+2^2}\right\} = f(s)$

$\therefore$ by shifting property

$\therefore L[e^{2t}.\cos^2 t] = \left\{\frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]\right\}_{s\to s-2}$

$= \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{(s-2)^2+4}\right]$

3) $(1+3t\,e^{2t})^2$

let $f(t) = (1+3t\,e^{2t})^2$, $\qquad (a+b)^2 = a^2+b^2+2ab$

apply L.T

$L[f(t)] = L[1 + 9t^2 e^{4t} + 6t\,e^{2t}]$

$L[f(t)] = L[1] + 9L[e^{4t}.t^2] + 6L[e^{2t}.t] \rightarrow ①$

consider $L[e^{4t}.t^2] = \left[\dfrac{2!}{s^3}\right]_{s\to s-4}$ $\quad$ (Replace s by s-4 & $L(t^2) = \frac{2!}{s^3}$)

$\therefore L[e^{4t}\cdot t^2] = \dfrac{2}{(s-4)^3}$

and consider. $L[t] = \dfrac{1}{s^2}$

$\therefore L[e^{2t}\cdot t] = \left[\dfrac{1}{s^2}\right]_{s \to s-2}$

$\therefore = \dfrac{1}{(s-2)^2}$

substitute these in ①.

$$L[f(t)] = \frac{1}{s} + 9\left[\frac{2}{(s-4)^3}\right] + 6\left[\frac{1}{(s-2)^2}\right]$$

$$= \frac{1}{s} + \frac{18}{(s-4)^3} + \frac{6}{(s-2)^2}$$

4) $\sinh at \cdot \sin at$

wkt $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

Let $f(t) = \sinh at \sin at = \left[\dfrac{e^{at} - e^{-at}}{2}\right] \sin at$

$$= \frac{1}{2}\left[e^{at} \sin at - e^{-at} \sin at\right]$$

apply L.T

$$L[f(t)] = \frac{1}{2}\left[L[e^{at} \sin at] - L[e^{-at} \sin at]\right]$$

$$= \frac{1}{2}\left[L[\sin at]_{s \to s-a} - L[\sin at]_{s \to s+a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{a}{s^2+a^2}\right)_{s \to s-a} - \left(\frac{a}{s^2+a^2}\right)_{s \to s+a}\right]$$

$$L[f(t)] = \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$$

6) $\cosh t \sin^3 2t$

wkt $\sin 3A = 3\sin A - 4\sin^3 A$ $\quad -4\sin^3\theta = \sin 3\theta - 3\sin\theta$

(Put $A = 2t$)

$\sin 3(2t) = 3\sin(2t) - 4\sin^3(2t)$ $\quad \sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$

$\sin 6t = 3\sin 2t - 4\sin^3 2t$

$\therefore \sin^3(2t) = \frac{1}{4}\left[3 \sin 2t - \sin 6t\right] \rightarrow (*)$

& wkt $\cosh t = \frac{e^t + e^{-t}}{2}$

$(*) \Rightarrow$

$L[\sin^3 2t] = \frac{1}{4}\left[3\, L[\sin 2t] - L[\sin 6t]\right]$

$= \frac{1}{4}\left[\frac{3(2)}{s^2+2^2} - \frac{6}{s^2+6^2}\right]$

$= \frac{1}{4}\left[\frac{6}{(s^2+4)} - \frac{6}{(s^2+36)}\right]$

$= \frac{1}{4}\left[\frac{(s^2+36)6 - 6(s^2+4)}{(s^2+4)(s^2+36)}\right] = \frac{1}{4}\left[\frac{6s^2+216-6s^2-24}{(s^2+4)(s^2+36)}\right]$

$= \frac{1}{4}\left[\frac{192}{(s^2+4)(s^2+36)}\right]$ 48

$L[\sin^3 2t] = \frac{48}{(s^2+4)(s^2+36)}$

Now $L[\cosh t . \sin^3 2t] = L\left[\frac{e^t + e^{-t}}{2} . \sin^3 2t\right]$

$= \frac{1}{2}\left[L(e^t \sin^3 2t) + L[e^{-t} \sin^3 2t]\right]$

$$= \frac{1}{2}\left[L[\sin^3 2t]_{s\to s-1} + L[\sin^3 2t]_{s\to s+1} \right]$$

$$= \frac{1}{\cancel{2}}\left\{ \frac{\cancel{48}\ 24}{((s-1)^2+4)[(s-1)^2+36]} + \frac{48}{[(s+1)^2+4][(s+1)^2+36]} \right\}$$

$\therefore L[\cosh t \,.\, \sin^3 2t]$

$$= \left[\frac{24}{(s^2-2s+5)(s^2-2s+37)} + \frac{24}{(s^2+2s+5)(s^2+2s+37)} \right]$$

Find the Laplace transform of the foll functions

1). $t \cos at$.

Let $f(t) = \cos at$.

$$L[f(t)] = L[\cos at] = \frac{s}{s^2+a^2} = f(s)$$

wkt. $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[f(s)]$

$n=1$

$$\Rightarrow L[t . f(t)] = (-1)\frac{d}{ds} f(s)$$

$$\therefore L[t\, f(t)] = \frac{-d}{ds}\left[\frac{s}{s^2+a^2}\right] \quad (\because \text{differentiate w.r.t } 's')$$

$$= -\left[\frac{(s^2+a^2)(1) - s(2s+0)}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] = -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$L[t . \cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

2) $t^2 \sin at$

Let $f(t) = \sin at$

$L[f(t)] = \frac{a}{s^2+a^2} = \bar{f}(s)$

$\left[L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \bar{f}(s) \right]$

consider $L[t^2 f(t)] = \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$ (diff 2 times wrt s)

$= \frac{d}{ds}\left\{\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]\right\}$ {apply $\frac{d}{dx}\left(\frac{u}{v}\right)$ rule} $= \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

$= \frac{d}{ds}\left[\frac{(s^2+a^2)(0) - a(2s)}{(s^2+a^2)^2}\right]$

$= \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$ (diff wrt 's')

$= -\left[\frac{(s^2+a^2)^2(+2a) - 2as\left(2(s^2+a^2)\,2s\right)}{(s^2+a^2)^4}\right]$

$= -\left[\frac{\cancel{(s^2+a^2)}\left[(s^2+a^2)(+2a) - 8as^2\right]}{(s^2+a^2)^{\cancel{4}\,3}}\right]$ (Take (s^2+a^2) as common factor)

$= -\left[\frac{+2s^2a + 2a^3 - 8as^2}{(s^2+a^2)^3}\right] = -\left[\frac{-6as^2 + 2a^3}{(s^2+a^2)^3}\right]$

$= \frac{2a\left[3s^2 - a^2\right]}{(s^2+a^2)^3}$ [∵ Take $2a$ as common factor)

$\therefore L[t^2 \sin at] = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

3) $t^3 \cosh t$

Let $f(t) = t^3 \cosh t = t^3\left[\frac{e^t + e^{-t}}{2}\right]$ $\quad \left\{\cosh t = \frac{e^t + e^{-t}}{2}\right.$

$$= \frac{1}{2}\left[e^t t^3 + e^{-t} t^3\right]$$

apply L.T

$$L[f(t)] = \frac{1}{2}\left[L[t^3]_{s \to s-1} + L[t^3]_{s \to s+1}\right]$$ (Property 1)

$$= \frac{1}{2}\left[\left[\frac{3!}{s^4}\right]_{s \to s-1} + \left[\frac{3!}{s^4}\right]_{s \to s+1}\right]$$

$$L[f(t)] = \frac{1}{2}\left\{\frac{6}{(s-1)^4} + \frac{6}{(s+1)^4}\right\}$$

4) $t^5 e^{4t} \cosh 3t$.

Let $f(t) = t^5 e^{4t} \cosh 3t = t^5 e^{4t}\left[\frac{e^{3t} + e^{-3t}}{2}\right]$

$$= \frac{1}{2}\left[e^{7t} t^5 + e^{t} t^5\right]$$

apply L.T.

$$L[f(t)] = \frac{1}{2}\left[L[e^{7t} t^5] + L[e^{t} t^5]\right]$$

$$= \frac{1}{2}\left[L[t^5]_{s \to s-7} + L[t^5]_{s \to s-1}\right\}$$ (property 1)

$$= \frac{1}{2}\left[\left(\frac{5!}{s^6}\right)_{s \to s-7} + \left(\frac{5!}{s^6}\right)_{s \to s-1}\right\}$$

$$= \frac{1}{2}\left[\frac{120}{(s-7)^6} + \frac{120}{(s-1)^6}\right]$$ (Take 120 as common & divide by 2)

$$L[f(t)] = 60\left[\frac{1}{(s-7)^6} + \frac{1}{(s-1)^6}\right]$$

5) $t \cdot e^{-2t} \sin 4t$

Let $f(t) = t e^{-2t} \sin 4t$

wkt $L[\sin 4t] = \dfrac{4}{s^2+16}$

$$L[e^{-2t} \sin 4t] = \left[\frac{4}{s^2+16}\right]_{s \to s+2} \quad \text{(property 1)}$$

$$= \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4+4s+16}$$

$$L[e^{-2t} \sin 4t] = \frac{4}{s^2+4s+20} = f(s)$$

Now

$$L[t(e^{-2t} \sin 4t)] = \frac{-d}{ds}\left[\frac{4}{s^2+4s+20}\right] \quad \text{(by property 2)}$$

diff w.r.t 's'

$$= -\left[\frac{(s^2+4s+20)(0) - 4(2s+4+0)}{(s^2+4s+20)^2}\right] \quad \text{by applying } \frac{u}{v} \text{ rule}$$

$$= -\left[\frac{0-8s-16}{(s^2+4s+20)^2}\right] = \frac{8s+16}{(s^2+4s+20)^2}$$

$$\therefore L[t e^{-2t} \sin 4t] = \frac{4[2s+4]}{(s^2+4s+20)^2}$$

Find the Laplace transform of the foll functions

i) $\frac{1-e^{-at}}{t} = \frac{f(t)}{t}$

Let $f(t) = 1-e^{-at} \Rightarrow L[f(t)] = L[1] - L[e^{-at}]$

$$= \frac{1}{s} - \frac{1}{s+a} = f(s)$$

wkt, $L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds$

$\Rightarrow L\left[\frac{1-e^{-at}}{t}\right] = \int_s^\infty \left[\frac{1}{s} - \frac{1}{s+a}\right] ds$ $\int 1/x\,dx = \log x$

$$= \left[\log s - \log(s+a)\right]_s^\infty$$

$$= \left\{\log\left[\frac{s}{s+a}\right]\right\}_s^\infty \quad \left(\log\frac{m}{n} = \log m - \log n\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{s}{s+a}\right) - \log\left(\frac{s}{s+a}\right)$$ (apply upper & lower limit)

$$= \lim_{s\to\infty} \log\left(\frac{\cancel{s}}{\cancel{s}(1+a/s)}\right) - \log\left(\frac{s}{s+a}\right)$$

[In first term, in the denominator take 's' as common]

If $s\to\infty \Rightarrow a/s \to 0$.

$$= \lim_{s\to\infty} \log\left(\frac{1}{1+a/s}\right) - \log\left(\frac{s}{s+a}\right)$$

$$= \log 1 - \log\left(\frac{s}{s+a}\right) \quad (\log 1 = 0)$$

$$= 0 - \left[\log s - \log(s+a)\right] \quad \log\frac{m}{n} = \log m - \log n$$

$$= -\log s + \log(s+a)$$

$$L\left[\frac{f(t)}{t}\right] = \log\left(\frac{s+a}{s}\right)$$

② $\dfrac{\cos at - \cos bt}{t} = \dfrac{f(t)}{t}$

Let $f(t) = \cos at - \cos bt$

$L[f(t)] = L[\cos at] - L[\cos bt] = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2} = f(s)$

wkt $L\left[\dfrac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds$

$= \int_s^\infty \left[\dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}\right] ds$

$= \int_s^\infty \dfrac{1}{2}\left[\dfrac{2s}{s^2+a^2} - \dfrac{2s}{s^2+b^2}\right] ds$ $\left\{ \int \dfrac{f'(x)}{f(x)}\,dx = \log f(x) \right.$

$= \dfrac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$

$= \dfrac{1}{2}\left[\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$ $\left(\log \dfrac{m}{n} = \log m - \log n\right)$

$= \dfrac{1}{2}\left\{\lim_{s\to\infty} \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right) - \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)\right\}$ (apply upper & lower limit)

$= \dfrac{1}{2}\left[\lim_{s\to\infty} \log\left(\dfrac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right) - \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)\right\}$

If $s\to\infty \Rightarrow a^2/s^2 \to 0$ & $b^2/s^2 \to 0$.

$\therefore L\left[\dfrac{f(t)}{t}\right] = \dfrac{1}{2}\left[\cancel{\lim_{s}} \log \overset{\to 0}{1} - \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)\right]$

$= \dfrac{1}{2}\left[-\log(s^2+a^2) + \log(s^2+b^2)\right]$ $\log \dfrac{m}{n} = \log m - \log n$

$= \dfrac{1}{2}\left[\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)\right]$ $(n \log m = \log m^n)$

$L\left[\dfrac{f(t)}{t}\right] = \log \sqrt{\dfrac{s^2+b^2}{s^2+a^2}}$

5) $\frac{\sin^2 t}{t} = \frac{f(t)}{t}$

$(\cos 2\theta = 1 - 2\sin^2\theta)$

Let $f(t) = \sin^2 t = \frac{1}{2}[1-\cos 2t]$

$L[f(t)] = \frac{1}{2}[L[1] - L[\cos 2t]]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = f(s)$

$L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds$.

$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{1}{2}\frac{2s}{s^2+4}\right) ds$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

$\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$

$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$

$\log m^n = n\log m$

$= \frac{1}{2}\left[\log\left[\frac{s}{\sqrt{s^2+4}}\right]\right]_s^\infty$

$= \frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{s}{\sqrt{s^2+4}}\right) - \log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]$

$= \frac{1}{2}\left[\lim_{s\to\infty}\log\frac{s}{\sqrt{s^2(1+4/s^2)}} - \log\frac{s}{\sqrt{s^2+4}}\right]$

$= \frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{\cancel{s}}{\cancel{s}\sqrt{(1+4/s^2)}}\right) - \log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]$

If $s\to\infty$ then $4/s^2 \to 0$

$= \frac{1}{2}\left[\cancel{\log 1}^{\to 0} - \log\left(\frac{s}{\sqrt{s^2+4}}\right)\right] = \frac{1}{2}\left[-(\log s - \log\sqrt{s^2+4})\right]$

$L\left[\frac{f(t)}{t}\right] = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$

4) $\frac{2\sin t \sin 5t}{t} = \frac{f(t)}{t}$

Let $f(t) = 2\sin t \sin 5t$

$= \frac{2}{2}\left[\cos(t-5t) - \cos(t+5t)\right]$

$\left[\sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]\right]$

$\therefore f(t) = \cos(-4t) - \cos(6t)$ $\quad [\cos(-\theta) = \cos\theta]$

$f(t) = \cos(4t) - \cos(6t)$

$\therefore L[f(t)] = L[\cos 4t] - L[\cos 6t]$

$L[f(t)] = \frac{s}{s^2+4^2} - \frac{s}{s^2+6^2} = f(s)$

wkt $L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds = \int_s^\infty \left(\frac{s}{s^2+16} - \frac{s}{s^2+36}\right) ds$

$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+16} - \frac{2s}{s^2+36}\right) ds$ $\quad$ (multiply & divide by 2)

$= \frac{1}{2}\left[\log(s^2+16) - \log(s^2+36)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{s^2+16}{s^2+36}\right) - \log\left(\frac{s^2+16}{s^2+36}\right)\right]$

$= \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{\cancel{s^2}(1+16/s^2)}{\cancel{s^2}(1+36/s^2)}\right) - \log\left(\frac{s^2+16}{s^2+36}\right)\right]$

If $s \to \infty \implies 16/s^2 \to 0$ & $36/s^2 \to 0$

$L\left[\frac{f(t)}{t}\right] = \frac{1}{2}\left[\log 1 - [\log(s^2+16) - \log(s^2+36)]\right]$

$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right] = \log\sqrt{\frac{s^2+36}{s^2+16}}$

5) $\frac{\sin at}{t} = \frac{f(t)}{t}$

let $f(t) = \sin at$

$L[f(t)] = L[\sin at] = \frac{a}{s^2+a^2} = f(s)$

$L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds = \int_s^\infty \left(\frac{a}{s^2+a^2}\right) ds$.

$= a \cdot \frac{1}{a}\left[\tan^{-1}(s/a)\right]_s^\infty$ $\qquad \int \frac{1}{x^2+a^2}dx = \tan^{-1}(x/a)$

$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$

$= \pi/2 - \tan^{-1}(s/a)$ $\qquad \cot^{-1}\theta = \pi/2 - \tan^{-1}\theta$

$L\left[\frac{\sin at}{t}\right] = \cot^{-1}(s/a)$

Evaluate the following integrals using L.T :-

1) $\int_0^\infty e^{0}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt$

$\Rightarrow$ wkt $\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$ (by defn)

$\Rightarrow \int_0^\infty e^{-st}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt = L\left[\frac{\cos 6t - \cos 4t}{t}\right] \rightarrow (*)$

consider

$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds$.

~~let~~ consider.

$L[\cos 6t - \cos 4t] = L[\cos 6t] - ~~\cos 4~~ L[\cos 4t]$

$= \frac{s}{s^2+6^2} - \frac{s}{s^2+4^2}$

$$L\left[\frac{\cos 6t-\cos 4t}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+36} - \frac{s}{s^2+16}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+36} - \frac{2s}{s^2+16}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+36) - \log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left[\frac{s^2+36}{s^2+16}\right]_s^\infty\right]$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{s^2+36}{s^2+16}\right) - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{\cancel{s^2}(1+36/s^2)}{\cancel{s^2}(1+16/s^2)}\right) - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\overset{\text{so}}{\cancel{\log}\ 1} - (\log(s^2+36) - \log(s^2+16)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

(*) $\Rightarrow$

$$\therefore \int_0^\infty e^{-st}\left[\frac{\cos 6t - \cos 4t}{t}\right] dt = \log\sqrt{\frac{s^2+16}{s^2+36}}$$

put $s=0$

$$\Rightarrow \int_0^\infty e^{0}\left[\frac{\cos 6t-\cos 4t}{t}\right] dt = \log\sqrt{\frac{0+16}{0+36}} = \log\sqrt{16/36}$$

$$\int_0^\infty \left[\frac{\cos 6t - \cos 4t}{t}\right] dt = \underline{\log(2/3)}$$

2) $\int_0^\infty \left(\frac{e^{-at} - e^{-bt}}{t}\right) dt$

first we find, $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

we are $L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\, ds$.

$L[e^{-at} - e^{-bt}] = L[e^{-at}] - L[e^{-bt}] = \frac{1}{s+a} - \frac{1}{s+b} = f(s)$

$\therefore L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds.$

$= \left[\log(s+a) - \log(s+b)\right]_s^\infty = \log\left(\frac{s+a}{s+b}\right)\Big]_s^\infty$

$= \lim_{s\to\infty} \log\left(\frac{s+a}{s+b}\right) - \log\left(\frac{s+a}{s+b}\right)$

$= \lim_{s\to\infty} \log\left(\frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right) - \log\left(\frac{s+a}{s+b}\right)$

If $s\to\infty \Rightarrow a/s$ & $b/s \to 0$

$\therefore L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \left[\cancel{\log 1}^{\,0} - \left[\log(s+a) - \log(s+b)\right]\right]$

$= \log\left(\frac{s+b}{s+a}\right)$

$\therefore \int_0^\infty e^{-st}\left[\frac{e^{-at} - e^{-bt}}{t}\right] dt = \log\left[\frac{s+b}{s+a}\right]$

Put $s=0$ then we have

$\int_0^\infty \left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \log(b/a)$

3) $\int_0^{\infty} \frac{e^{-t} \sin t}{t} \, dt$

wkt $L[\sin t] = \frac{1}{s^2+1}$

$L[e^{-t} \sin t] = \left[\frac{1}{s^2+1}\right]_{s \to s+1}$

$= \frac{1}{(s+1)^2 + 1} = f(s)$

wkt

$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} f(s) \, ds$

$L\left[\frac{e^{-t} \sin t}{t}\right] = \int_s^{\infty} \left(\frac{1}{(s+1)^2+1}\right) ds = \tan^{-1}(s+1)\Big]_s^{\infty}$

$= \tan^{-1}(\infty) - \tan^{-1}(s+1)$

$= \pi/2 - \tan^{-1}(s+1)$

$\boxed{\cot\theta = \pi/2 - \tan\theta}$

$L\left[\frac{e^{-t} \sin t}{t}\right] = \cot^{-1}(s+1)$

by defn.

$\int_0^{\infty} e^{-st} \cdot \frac{e^{-t} \sin t}{t} \cdot dt = \cot^{-1}(s+1)$

put $s = 0$

$\int_0^{\infty} \frac{e^{-t} \sin t}{t} \, dt = \cot(1)$

$= \pi/4$

4) $\frac{\sin^2 t}{t}$

Soln. $f(t) = \sin^2 t = \frac{1}{2}[1-\cos 2t]$ $\quad \cos 2\theta = 2\sin^2 t -$

$L[f(t)] = \frac{1}{2}[L[1] - L[\cos 2t]]$

$L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$\therefore L\left[\frac{f(t)}{t}\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$ $\quad = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{1}{2}\frac{2s}{(s^2+4)}\right) ds$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$ $\quad \int \frac{f'(x)}{f(x)} = \log f(x)$

$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$

$= \frac{1}{2}\left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty =$

$= \frac{1}{2}\left\{\lim_{s\to\infty} \frac{s}{\sqrt{s^2(1+4/s^2)}} - \log\frac{s}{\sqrt{s^2+4}}\right]$

$= \frac{1}{2}\left[\lim_{s\to\infty} \frac{\cancel{s}}{\cancel{s}\sqrt{1+4/s^2}} - \log\frac{s}{\sqrt{s^2+4}}\right]$

$= \frac{1}{2}\left[\cancel{\log(1)}^{\to 0} - \log\frac{s}{\sqrt{s^2+4}}\right]$ $\quad \left\{ \begin{array}{l} \text{As} \\ s\to\infty \Rightarrow 4/s^2 \to 0 \end{array}\right]$

$= \frac{1}{2}\left[-(\log s - \log(\sqrt{s^2+4}))\right]$

$= \frac{1}{2}\left[\log\sqrt{s^2+4} - \log s\right]$

$\Rightarrow L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}$

* Find the Laplace transform of the foll functions

1) $\int_0^t \sinh at \, \sin at \, dt$

Let $f(t) = \sinh at \cdot \sin at$

$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

$= \frac{(e^{at} - e^{-at})}{2} \cdot \sin at$

$= \frac{1}{2}\left[e^{at} \sin at - e^{-at} \sin at\right]$

$L[f(t)] = \frac{1}{2}\left[\left(\frac{a}{s^2+a^2}\right)_{s \to s-a} - \left(\frac{a}{s^2+a^2}\right)_{s \to s+a}\right]$

$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$

$= \frac{a}{2}\left[\frac{1}{s^2+a^2-2as+a^2} - \frac{a}{s^2+a^2+2as+a^2}\right]$

$= \frac{a}{2}\left[\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right]$

$= \frac{a}{2}\left[\frac{\cancel{s^2}+\cancel{2a^2}+2as - \cancel{s^2}-\cancel{2a^2}+2as}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$

$= \frac{a}{\cancel{2}}\left[\frac{\overset{2}{\cancel{4}}as}{(s^2+2a^2)^2-(2as)^2}\right]$

$(a+b)(a-b) = a^2-b^2$

$L[f(t)] = \frac{2a^2 s}{s^4+4a^4} = f(s)$

we have

$L\left[\int_0^t f(t) \, dt\right] = \frac{f(s)}{s}$

$\therefore L\left[\int_0^t \sinh at \, \sin at \, dt\right] = \frac{2a^2 \cancel{s}}{\cancel{s}(s^4+4a^4)} = \frac{2a^2}{s^4+4a^4}$

2) $\int_0^t t\cos at\,dt$

let $f(t) = t\cos at$

wkt ~~L~~ $L[\cos at] = \frac{s}{s^2+a^2}$

$\because L[t\,f(t)] = -\frac{d}{ds} f(s)$

$L[t.\cos at] = -\frac{d}{ds}\left[\frac{s}{(s^2+a^2)}\right]$

$= -\left[\frac{(s^2+a^2) - s(2s+0)}{(s^2+a^2)^2}\right] = \frac{-s^2-a^2+2s^2}{(s^2+a^2)^2}$

$L[t.\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$

We have, $L\left[\int_0^t f(t)\,dt\right] = \frac{f(s)}{s}$

$\Rightarrow L\left[\int_0^t t.\cos at\,dt\right] = \frac{(s^2-a^2)}{s(s^2+a^2)^2}$

3) $\int_0^t e^{2t}\frac{\sin at}{t}\,dt$

let $f(t) = e^{2t}\frac{\sin at}{t}$

$L[f(t)] = L\left[\frac{\sin at}{t}\right]_{s\to s-2}$

let $L[\sin at] = \frac{a}{s^2+a^2}$

$\therefore L\left[\frac{\sin at}{t}\right] = \int_s^\infty \left(\frac{a}{s^2+a^2}\right)ds$

$= a\left(\frac{1}{a}\tan^{-1}(s/a)\right]_s^\infty$

$= \frac{a}{a}\left[\tan^{-1}(\infty) - \tan^{-1}(s/a)\right]$

$= \pi/2 - \tan^{-1}(s/a)$

$= \cot^{-1}(s/a)$

$$\therefore L\left[\frac{\sin at}{t}\right] = \cot^{-1}(s/a)$$

$$\therefore L\left[e^{2t}\cdot\frac{\sin at}{t}\right] = \left[\cot^{-1}(s/a)\right]_{s\to s-2}$$

$$= \cot^{-1}\left(\frac{s-2}{a}\right) = f(s)$$

$\because$ we have, $$L\left[\int_0^t f(t)\,dt\right] = \frac{f(s)}{s} = \frac{1}{s}\cot^{-1}\left(\frac{s-2}{a}\right)$$

4) $e^{-4t}\cdot\int_0^t t\sin 3t\,dt$

First we shall find. $L[t\sin 3t]$

$$\Rightarrow L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$$

$$\therefore L[t\sin 3t] = -\frac{d}{ds}\left[\frac{3}{s^2+9}\right] = -\left[\frac{(s^2+9)\overset{\text{so}}{(0)} - 3(2s)}{(s^2+9)^2}\right]$$

$$= \frac{6s}{(s^2+9)^2}$$

we have $$L\left[\int_0^t f(t)\,dt\right] = \frac{f(s)}{s}$$

$$\therefore L\left[\int_0^t t\sin 3t\,dt\right] = \frac{1}{\cancel{s}}\cdot\frac{6\cancel{s}}{(s^2+9)^2} = \frac{6}{(s^2+9)^2}$$

$$\therefore L\left[e^{-4t}\int_0^t t\sin 3t\cdot dt\right] = \left[\frac{6}{(s^2+9)^2}\right]_{s\to s+4} = \frac{6}{\left[(s+4)^2+9\right]^2}$$

Expanding

$$= \frac{6}{(s^2+8s+25)^2}$$

Laplace Transform of Periodic function :-

Defn :- A function $f(t)$ is said to be periodic function of period $T > 0$, if $f(t+nt) = f(t)$ where $n = 1, 2, 3 \ldots$

Ex :- $\sin t$, $\cos t$, are periodic function of period 2π.

$\because \sin(t + 2n\pi) = \sin t$, $\cos(t + 2n\pi) = \cos t$.

Statement :- If $f(t)$ is a periodic function of period T, Then

$$\boxed{L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt}$$

Examples

1) If $f(t) = t^2$, $0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$, find $L[f(t)]$.

soln :- Here, $f(t)$ is a periodic function of period 2.

$\therefore T = 2 \Rightarrow f(t) = t^2$.

we know that, $L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$\Rightarrow L[f(t)] = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} \cdot t^2\, dt = \frac{1}{1 - e^{-2s}} \int_0^2 t^2 e^{-st}\, dt$$

Apply Bernoulli's rule of Integration by parts

$$L[f(t)] = \frac{1}{1 - e^{-2s}} \left[t^2 \cdot \frac{e^{-st}}{-s} - (2t) \frac{e^{-st}}{(-s)(-s)} + 2 \frac{e^{-st}}{(-s)^3} \right]_0^2$$

$$\left\{ \int uv = u \int v - u' \iint v + u''' \iiint v - + \ldots \right\}$$

$$L[f(t)] = \left\{ \frac{-1}{s}\left(4e^{-2s}-0\right) - \frac{2}{s^2}\left[2e^{-2s}-0\right] - \frac{2}{s^3}\left[e^{-2s}-1\right] \right\}$$

$$= \frac{2}{s^3(1-e^{-2s})}\left[-2s^2e^{-2s} - 2se^{-2s} - e^{-2s} + 1\right]$$ (Take common $2/s^3$)

$$L[f(t)] = \frac{2}{s^3(1-e^{-2s})}\left[1-(2s^2+2s+1)e^{-2s}\right]$$

2) Given $f(t) = \begin{cases} E & 0 < t < a/2 \\ -E & a/2 < t < a. \end{cases}$

where $f(t+a) = f(t)$, show that $L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$

sol → The given function is period with period $T = a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-sa}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-sa}}\left\{\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right\}$$

$$= \frac{1}{1-e^{-sa}}\left[\int_0^{a/2} e^{-st}\cdot E\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right\}$$

$$= \frac{1}{1-e^{-sa}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{a/2} + \left[\frac{e^{-st}}{s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left[-\left(e^{-st}\right)_0^{a/2} + \left[e^{-st}\right]_{a/2}^{a}\right]$$

$$= \frac{E}{s(1-e^{-as})}\left[-e^{-as/2}+1+e^{-as}-e^{-as/2}\right]$$

$$= \frac{E}{s(1-e^{-as})}\left[1-2e^{-as/2}+e^{-as}\right]$$

$$= \frac{E(1-e^{-as/2})^2}{s(1-e^{-as})}$$

$(a-b)^2 = a^2-2ab+b^2$

$$L[f(t)] = \frac{E(1-e^{-as/2})^{\cancel{2}}}{s\cancel{(1-e^{-as/2})}(1+e^{-as/2})} = \frac{E(1-e^{-as/2})}{s(1+e^{-as/2})}$$

multiply both the numerator & denominator by $e^{as/4}$.

$$L[f(t)] = \frac{E\left[e^{as/4}-e^{-as/4}\right]}{s\left[e^{as/4}+e^{-as/4}\right]}$$

$$= \frac{E}{s}\,\frac{\cancel{2}\sinh(as/4)}{\cancel{2}\cosh(as/4)}$$

$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

$\Rightarrow e^{\theta}-e^{-\theta} = 2\sinh\theta$

$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$

$e^{\theta}+e^{-\theta} = 2\cosh\theta$

$$L[f(t)] = \frac{E}{s}\tanh(as/4)$$

3) A periodic function of period $2\pi/\omega$ is defined by

$$f(t) = \begin{cases} E \sin \omega t, & 0 \le t < \pi/\omega \\ 0, & \pi/\omega \le t < 2\pi/\omega \end{cases}$$

where E and ω are constants.

Show that. $L[f(t)] = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

sol: wkt, for a periodic function $f(t)$,

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad \text{Here } T = \frac{2\pi}{\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)\, dt.$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} (E \sin \omega t)\, dt + \cancel{\int_{\pi/\omega}^{2\pi/\omega} e^{-st}(0)\, dt}^{\to 0} \right] \to (*)$$

wkt, $\displaystyle\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$

$(*) \Rightarrow$

$$L[f(t)] = \frac{E}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{(-s)^2+\omega^2} \left[-s \sin \omega t - \omega \cos \omega t \right]_0^{\pi/\omega} \right]$$

$$= \frac{-E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})} \left[e^{-s\pi/\omega} (s \overbrace{\sin \pi}^{0} + \omega \overbrace{\cos \pi}^{(-1)}) - e^0 (s \overbrace{\sin 0}^{0} + \omega \cos(0)) \right]$$

$$= \frac{-E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})} \left[-\omega e^{-s\pi/\omega} - \omega \right]$$

$$= \frac{E\omega \left[1 + e^{-\pi s/\omega}\right]}{(s^2+\omega^2)\left[1 - e^{-2\pi s/\omega}\right]}$$

(or) $L[f(t)] = \dfrac{E\omega\left(1+e^{-\pi s/\omega}\right)}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)\left(1+e^{-\pi s/\omega}\right)}$

$$L[f(t)] = \frac{E\omega}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)}$$

4) Find $L[f(t)]$, where $f(t) = e^{-t}$, $0<t<1$.

solⁿ :- The period of $f(t) = 1-0 = 1$. ie, $T = 1$

wkt, $L[f(t)] = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} \cdot f(t) \, dt$

$$= \frac{1}{1-e^{-s}} \int_0^1 e^{-st} \cdot e^{-t} \, dt = \frac{1}{1-e^{-s}} \int_0^1 e^{-(s+1)t} \, dt$$

$$= \frac{1}{1-e^{-s}} \left[\frac{e^{-(s+1)t}}{-(s+1)} \right]_0^1 = \frac{1}{1-e^{-s}} \left[\frac{e^{-(s+1)}}{-(s+1)} - \frac{e^{-(s+1)0}}{-(s+1)} \right]$$

$$L[f(t)] = \frac{1-e^{-(s+1)}}{(1-e^{-s})(s+1)}$$

Unit step function [Heaviside function]

Definition :- The unit step function $u(t-a)$ or Heaviside function $H(t-a)$ is defined as follows,

$$u(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$

where 'a' is a positive constant.

Properties

1) $L[u(t-a)] = \frac{e^{-as}}{s}$

2) $L[f(t-a)u(t-a)] = e^{-as} \bar{f}(s)$ where $L[f(t)] = \bar{f}(s)$

Note :→

1) If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t) & t > a \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a)$

2) If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a)$
$\quad + [f_3(t) - f_2(t)]\, u(t-b).$

working procedure

To find $L[f(t)\,u(t-a)]$ where $F(t)$ is a polynomial in 't'.

* Let $F(t) = f(t-a)$ which implies that $F(t+a) = f(t)$.
* Replace t by $(t+a)$ to obtain $f(t)$.
* Find $L[f(t)] = f(s)$
* $L[F(t)\,u(t-a)] = e^{-as} f(s)$ by property ②.

Examples

Find the Laplace transform of the following functions

1). $(3t^2+4t+5)\,u(t-3)$

Let $f(t-3) = 3t^2+4t+5$

Replace t by $t+3$.

$$f(t) = 3(t+3)^2 + 4(t+3) + 5.$$

$$= 3[t^2+9+6t] + 4t + 12 + 5$$

$$= 3t^2 + \underline{27} + 18t + 4t + \underline{17}$$

$$f(t) = 3t^2 + 22t + 44$$

Apply L.T on both side

$$L[f(t)] = 3.L[t^2] + 22\,L[t] + 44\,L[1]$$

$$= 3.\frac{2!}{s^3} + 22.\frac{1!}{s^2} + 44\frac{1}{s}$$

$$L[f(t)] = \frac{6}{s^3} + \frac{22}{s^2} + \frac{44}{s}.$$

w.k.t $L[f(t-a)\,u(t-a)] = e^{-as} f(s)$ where $L[f(t)] = f(s)$

$\Rightarrow L[f(t-3)\,u(t-3)] = e^{-3s}\, L[f(t)]$

$L[(3t^2+4t+5)\,u(t-3)] = e^{-3s}\left[\frac{6}{s^3} + \frac{22}{s^2} + \frac{44}{s}\right]$

2) $[e^{t-1} + \sin(t-1)]\,u(t-1)$

Let $f(t-1) = e^{t-1} + \sin(t-1)$

Replace t by $t+1$

$\Rightarrow f(t) = e^t + \sin t$

Apply L.T, $\Rightarrow L[f(t)] = L[e^t] + L[\sin t]$

$= \frac{1}{s-1} + \frac{1}{s^2+1}$

$\Rightarrow L[f(t-1)\,u(t-1)] = e^{-s}\, L[f(t)]$

$L[e^{t-1} + \sin(t-1)\,u(t-1)] = e^{-s}\left[\frac{1}{s-1} + \frac{1}{s^2+1}\right]$

3) $\sin t\ u(t-\pi)$

Let $f(t-\pi) = \sin t$

Replace t by $t+\pi$

$f(t) = \sin(t+\pi) = -\sin t$

$L[f(t)] = -L[\sin t] = -\frac{1}{s^2+1} = f(s)$

w.k.t $L[f(t-\pi)\,u(t-\pi)] = e^{-\pi s}\, f(s)$

$L[\sin t \cdot u(t-\pi)] = e^{-\pi s}\left(-1/s^2+1\right)$

4) $(1-e^{2t})\, u(t+1)$

Let $f(t+1) = 1-e^{2t}$

Replace t by $t-1$

$$f(t) = 1-e^{2(t-1)} = 1-e^{2t-2}$$

$$f(t) = 1-e^{-2}.e^{2t}$$

Apply L.T, $\Rightarrow L[f(t)] = L[1]-e^{-2}.L[e^{2t}]$

$$= \left(\frac{1}{s}\right)-e^{-2}\left(\frac{1}{s-2}\right)$$

We've $L[f(t-a)\,u(t-a)] = e^{-as} f(s)$.

Here $a=-1$,

$$L[f(t+1)\,u(t+1)] = e^{s}\left[\frac{1}{s} - \frac{e^{-2}}{s-2}\right]$$

$$L[(1-e^{2t})\,u(t+1)] = e^{s}\left[\frac{1}{s} - \frac{1}{e^{2}(s-2)}\right]$$

---

5) $(t^2-6t+9)\,e^{-(t-3)}\,u(t-3)$

Let $f(t-3) = (t^2-6t+9)\,e^{-(t-3)}$

$(a-b)^2 = a^2-2ab+b^2$

$$f(t-3) = (t-3)^2 e^{-(t-3)}$$

Replace t by $t+3$.

$$f(t) = t^2 e^{-t}$$

apply L.T

$$L[f(t)] = L[t^2 e^{-t}] \rightarrow (*)$$

$$L[t^2] = \frac{2!}{s^3}$$

$$L[e^{-t}\,t^2] = \left[\frac{2!}{s^3}\right]_{s\to s+1}$$

$$= \frac{2}{(s+1)^3}$$

$(*) \Rightarrow$

$$L[f(t)] = \frac{2}{(s+1)^3}$$

$\therefore$ we have $(a=+3)$

$$L[f(t-3)\,u(t-3)] = e^{-3s} f(s)$$

$$L[(t^2-6t+9)\,e^{-(t-3)}\,u(t-3)]$$

$$= e^{-3s}\frac{2}{(s+1)^3}$$

Express the following functions in terms of Heaviside unit step function and hence find their Laplace transform:-

1) $f(t) = \begin{cases} t & , 0<t<4 \\ 5 & , t>4. \end{cases}$

soln: By a property,

$f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a)$

$\Rightarrow f(t) = t + [5-t)\, u(t-4)$

$L[f(t)] = L[t] + L[(5-t)\, u(t-4)] \longrightarrow (1)$

wkt $L[t] = 1/s^2$

Let $F(t-4) = 5-t$

Replace t by $t+4$

$F(t) = 5-(t+4) = 5-t-4 = 1-t.$

$F(t) = 1-t$

apply L.T $\Rightarrow L[F(t)] = L[1] - L[t]$

$= 1/s - 1/s^2 = F(s)$

But $L[F(t-4)\, u(t-4)] = e^{-4s} F(s)$ [Here $a=4$]

$L[(5-t)\, u(t-4)] = e^{-4s}(1/s - 1/s^2)$

$\therefore$ from (1)

$$L[f(t)] = \frac{1}{s^2} + e^{-4s}\left[\frac{1}{s} - \frac{1}{s^2}\right]$$

2) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

sol: $f(t) = \cos t + (\sin t - \cos t)\, u(t-\pi)$

$L[f(t)] = L[\cos t] + L[(\sin t - \cos t)\, u(t-\pi)] \rightarrow ①$

wkt $L[\cos t] = \dfrac{s}{s^2+1}$

Let $F(t-\pi) = \sin t - \cos t$.

Replace t by $t+\pi$.

$F(t) = \sin(t+\pi) - \cos(t+\pi)$

$F(t) = -\sin t - (-\cos t) = -\sin t + \cos t$

$L[F(t)] = -L[\sin t] + L[\cos t]$

$= \dfrac{-1}{s^2+1} + \dfrac{s}{s^2+1} = \dfrac{s-1}{s^2+1} = F(s)$

But $L[F(t-\pi)\, u(t-\pi)] = e^{-\pi s} F(s)$

$= e^{-\pi s}\left(\dfrac{s-1}{s^2+1}\right)$

substitute these in ①

$L[f(t)] = \dfrac{s}{s^2+1} + \dfrac{e^{-\pi s}(s-1)}{s^2+1}$

$L[f(t)] = \dfrac{s + e^{-\pi s}(s-1)}{s^2+1}$

3) $f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ t, & 1 < t \leq 2 \\ t^2, & t > 2 \end{cases}$ (37)

Soln :- $f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a) + [f_3(t) - f_2(t)]\, u(t-b)$

$\Rightarrow f(t) = 1 + [(t-1)\, u(t-1)] + (t^2 - t)\, u(t-2) \longrightarrow (1)$

Let $\underset{F(s)}{L[f(t)]} = L[1] + \underline{L[(t-1)\, u(t-1)]} + \underline{L[(t^2-t)\, u(t-2)]}$

wkt $L[1] = 1/s$.

Let
$F(t-1) = t-1$
Replace t by $t+1$
$F(t) = t$

$L[F(t)] = L[t]$

$L[F(t)] = 1/s^2 = F(s)$

But $(a=1)$

$L[F(t-1)u(t-1)] = e^{-s} F(s)$

$L[(t-1)u(t-1)] = e^{-s} \frac{1}{s^2}$

Let
$G(t-2) = t^2 - t$
Replace t by $t+2$
$G(t) = (t+2)^2 - (t+2)$
$= t^2 + 4 + 4t - t - 2$
$G(t) = t^2 + 3t + 2$

$L[G(t)] = L[t^2] + 3L[t] + 2L[1]$

$= \frac{2!}{s^3} + 3 \frac{1!}{s^2} + 2 \frac{1}{s}$

$L[G(t)] = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s} = G(s)$

$\therefore L[G(t-2)u(t-2)] = e^{-2s} G(s)$

$= e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$

$\therefore$ from (1)

$$L[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$$

④ $f(t) = \begin{cases} \cos t, & 0 < t \le \pi, \\ 1 & \pi < t \le 2\pi \\ \sin t, & t > 2\pi \end{cases}$

Soln.. $f(t) = \cos t + (1-\cos t)\, u(t-\pi) + (\sin t - 1)\, u(t-2\pi)$

$$L[f(t)] = L[\cos t] + L[(1-\cos t)u(t-\pi)] + L[(\sin t - 1)u(t-2\pi)] \rightarrow ①$$

wkt $L[\cos t] = \dfrac{s}{s^2+1}$.

To find $L[(1-\cos t)u(t-\pi)]$, & $L[(\sin t -1)u(t-2\pi)]$

Let

$F(t-\pi) = 1 - \cos t$

Replace t by $t+\pi$

$F(t) = 1 - \cos(t+\pi)$

$= 1 - (-\cos t)$

$F(t) = 1 + \cos t$

apply L.T

$L[F(t)] = L[1] + L[\cos t]$

$= 1/s + \dfrac{s}{s^2+1} = F(s)$

But $(\because a = \pi)$

$L[F(t-\pi)\, u(t-\pi)] = e^{-\pi s} F(s)$

$L[(1-\cos t)\, u(t-\pi)] = e^{-\pi s}\left(\dfrac{1}{s} + \dfrac{s}{s^2+1}\right)$

Let

$g(t-2\pi) = \sin t - 1$

Replace t by $t + 2\pi$

$G(t) = \sin(t+2\pi) - 1$

$G(t) = \sin t - 1$.

apply L.T

$L[G(t)] = L[\sin t] - L[1]$

$= \dfrac{1}{s^2+1} - \dfrac{1}{s} = G(s)$

But (Here $a = 2\pi$)

$L[g(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\, G(s)$

$L[(\sin t - 1)\, u(t-2\pi)] = e^{-2\pi s}\left(\dfrac{1}{s^2+1} - \dfrac{1}{s}\right)$

$\therefore$ Eqn ① becomes

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)$$

5) $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \geq 2\pi \end{cases}$

Soln :- $f(t) = \sin t + (\sin 2t - \sin t)\, u(t-\pi) + (\sin 3t - \sin 2t)\, u(t-2\pi)$ $\longrightarrow$ ①

wkt

$L[f(t)] = L[\sin t] + L[(\sin 2t - \sin t)\, u(t-\pi)] + L[(\sin 3t - \sin 2t)\, u(t-2\pi)] \longrightarrow$ ①

wkt $L[\sin t] = 1/s^2+1$

Now, we find $L[(\sin 2t - \sin t) u(t-\pi)]$, $L[(\sin 3t - \sin 2t) u(t-2\pi)]$

Let

$F(t-\pi) = \sin 2t - \sin t.$

Replace t by $t+\pi$

$F(t) = \sin(2t+2\pi) - \sin(t+\pi)$

$= \sin 2t - (-\sin t)$

$F(t) = \sin 2t + \sin t$

$L[F(t)] = L[\sin 2t] + L[\sin t]$

$L[F(t)] = \dfrac{2}{s^2+4} + \dfrac{1}{s^2+1} = F(s)$

But

$\therefore L[F(t-\pi) u(t-\pi)] = e^{-\pi s} F(s)$

$L[(\sin 2t - \sin t) u(t-\pi)]$

$= e^{-\pi s}\left(\dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}\right)$

Let

$G(t-2\pi) = \sin 3t - \sin 2t$

Replace t by $t+2\pi$

$G(t) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$

$= \sin 3t - \sin 2t$

$L[G(t)] = L[\sin 3t] - L[\sin 2t]$

$L[G(t)] = \dfrac{3}{s^2+9} - \dfrac{2}{s^2+4} = G(s)$

But

$L[G(t-2\pi) u(t-2\pi)] = e^{-2\pi s} G(s)$

$L[(\sin 3t - \sin 2t) u(t-2\pi)]$

$= e^{-2\pi s}\left(\dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}\right)$

$\therefore$ From eqn ①

$\Rightarrow L[f(t)] = \dfrac{1}{s^2+1} + e^{-\pi s}\left(\dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}\right) + e^{-2\pi s}\left(\dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}\right)$

(6) $f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & t>2 \end{cases}$

Soln:- $f(t) = 0 + (t-1)\,u(t-1) + (\underset{1-t+1}{1-(t-1)})\,u(t-2)$

$= 0 + (t-1)\,u(t-1) - (t-2)\,u(t-2)$

$L[f(t)] = L[(t-1)\,u(t-1)] - L[(t-2)\,u(t-2)] \longrightarrow (1)$

Let

$F(t-1) = t-1$

Replace t by $t+1$

$F(t) = t$

$L[F(t)] = L[t]$

$L[F(t)] = \frac{1}{s^2} = F(s)$

Let

$G(t-2) = t-2$

Reply t by $t+2$.

$G(t) = t$

$L[G(t)] = L[t]$

$= 1/s^2 = G(s)$

But

$L[F(t-1)\,u(t-1)] = e^{-s} F(s)$

$L[(t-1)\,u(t-1)] = e^{-s} \frac{1}{s^2}$

$L[G(t-2)\,u(t-2)] = e^{-2s} G(s)$

$L[(t-2)\,u(t-2)] = e^{-2s} \frac{1}{s^2}$

(1) $\Rightarrow$

$L[f(t)] = e^{-s}\frac{1}{s^2} - e^{-2s}\frac{1}{s^2} = \frac{e^{-s} - e^{-2s}}{s^2}$

Unit Impulse function

The unit impulse function (or) the dirac delta function $\delta(t-a)$ is defined as follows.

$$\delta(t-a) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a) \; ; \; a \geq 0$$

where $\delta_\varepsilon(t-a) = \begin{cases} 1/\varepsilon & \text{if } a \leq t \leq a+\varepsilon \\ 0 & \text{otherwise.} \end{cases}$

Property

* $L[\delta(t-a)] = e^{-as}$

In particular if $a=0$, $L[\delta(t)] = 1$.

Examples

1) Find $L[2\delta(t-1) + 3\delta(t-2) + 4\delta(t+3)]$

soln:- wkt $L[\delta(t-a)] = e^{-as}$

$$\therefore 2L[\underset{a=1}{\delta(t-1)}] + 3L[\underset{a=2}{\delta(t-2)}] + 4L[\underset{a=-3}{\delta(t+3)}]$$

$$= 2e^{-s} + 3e^{-2s} + 4e^{3s}$$

2). Find $L[\cosh 3t \, \delta(t-2)]$

soln: wkt $\cosh 3t \, \delta(t-2) = \left(\frac{e^{3t}+e^{-3t}}{2}\right)\delta(t-2)$

$\Rightarrow$ apply L.T $= \frac{1}{2}[e^{3t}\delta(t-2) + e^{-3t}\delta(t-2)]$

$$L[\cosh 3t \cdot \delta(t-2)] = \frac{1}{2}\left[L[e^{3t}\delta(t-2)] + L[e^{-3t}\delta(t-2)]\right]$$

$$= \frac{1}{2}\left\{ L[\delta(t-2)]_{s \to s-3} + L[\delta(t-2)]_{s \to s+3} \right]$$

$\longrightarrow$ (a)

(7) obtain the Laplace transform of $e^{-t}[1-u(t-2)]$

soln: $f(t) = e^{-t}[1-u(t-2)] = e^{-t} - e^{-t}u(t-2)$

$$L[f(t)] = L[e^{-t}] - L[e^{-t}u(t-2)] \rightarrow (1)$$

wkt $L[e^{-t}] = \frac{1}{s+1}$

& let

$F(t-2) = e^{-t}$

Replace t by $t+2$.

$$F(t) = e^{-(t+2)} = e^{-t-2} = e^{-t} \cdot e^{-2}$$

$$L[F(t)] = e^{-2} L[e^{-t}] = e^{-2} \frac{1}{(s+1)} = F(s)$$

$$L[F(t-2)u(t-2)] = e^{-2s} \cdot F(s)$$

$$= e^{-2s} \frac{e^{-2}}{s+1}$$

eqn (1) $\Rightarrow$

$$L[f(t)] = \frac{1}{s+1} - \frac{e^{-2s}e^{-2}}{s+1} = \frac{1-e^{-2} \cdot e^{-2s}}{s+1}$$

$$\boxed{L[f(t)] = \frac{1-e^{-2(s+1)}}{(s+1)}}$$

$= \frac{1}{2}\left[\left(e^{-2s}\right)_{s\to s-3} + \left(e^{-2s}\right)_{s\to s+3}\right]$

$= \frac{1}{2}\left[e^{-2(s-3)} + e^{-2(s+3)}\right] = \frac{1}{2}\left[e^{-2s+6} + e^{-2s}e^{-6}\right]$

$\therefore L[\cosh 3t\ \delta(t-2)] = \frac{e^{-2s}}{2}\left[e^{6}+e^{-6}\right]$

$= \cosh 6 .\ e^{-2s}$

$\cosh 6 = \frac{e^{6}+e^{-6}}{2}$

3) Find $L[t^4\ \delta(t-3)]$

sol :- wkt $L[\delta(t-a)] = e^{-as}$.

$\therefore L[\delta(t-3)] = e^{-3s}$ $(\because a=3)$

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$L[t^4\ \delta(t-3)] = (-1)^4 \frac{d^4}{ds^4}\left[e^{-3s}\right]$

$= (-1)^4 \left[(-3)^4\ e^{-3s}\right]$

$L[t^4\ \delta(t-3)] = 81\ e^{-3s}$

$\frac{d^3}{ds^3}\frac{d}{ds}\left(e^{-3s}\right)$

$(-3)\frac{d^2}{ds^2}\left((-3)\ e^{-3s}\right)$

$(-3)^3 \frac{d}{ds}\left(e^{-3s}\right)$

$= (-3)^4\ e^{-3s}$.

4) Find $L[(t-1)^2\delta(t-a)]$

sol => $(t-1)^2\ \delta(t-a) = (t^2-2t+1)\ \delta(t-a)$

$= t^2\ \delta(t-a) - 2\,t\ \delta(t-a) + \delta(t-a)$

Apply L.T

$L[(t-1)^2\delta(t-a)] = L[t^2\delta(t-a)] - 2L[t\ \delta(t-a)] + L[\delta(t-a)]$

$= (-1)^2 \frac{d^2}{ds^2}\left(e^{-as}\right) - 2\,(-1)^1 \frac{d}{ds}\left(e^{-as}\right) + e^{-as}$

$$= a^2 e^{-as} - 2a e^{-as} + e^{-as}$$

$$\therefore L[(t-1)^2 \delta(t-a)] = (a^2 - 2a + 1)\, e^{-as}$$
$$= (a-1)^2\, e^{-as}$$

$$\frac{d^2}{ds^2}(e^{-as}) = \frac{d}{ds}\frac{d}{ds}(e^{-as})$$
$$= (-a)\frac{d}{ds}(e^{-as})$$
$$= (-a)(-a)\cdot e^{-as}$$
$$= (-a)^2 e^{-as}$$

5) Find $L\left[\frac{2\delta(t-3) + 3\delta(t-2)}{t}\right]$

consider, $L[2\delta(t-3) + 3\delta(t-2)] = 2L[\delta(t-3)] + 3L[\delta(t-2)]$
$$= 2e^{-3s} + 3e^{-2s}.$$

$$\therefore L\left[\frac{2\delta(t-3)+3\delta(t-2)}{t}\right] = \int_s^\infty (2e^{-3s} + 3e^{-2s})\,ds.$$

{Here we use $L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds.$}

$$\therefore = \left[\frac{2e^{-3s}}{-3} + \frac{3e^{-2s}}{-2}\right]_s^\infty$$

$$= \left[\cancel{\frac{-2}{3}e^{-\infty}} + \cancel{\frac{3}{-2}e^{-\infty}}\right] - \left[\frac{-2}{3}e^{-3s} - \frac{3}{+2}e^{-2s}\right] \qquad (e^{-\infty} = 0)$$

$$\therefore L\left[\frac{2\delta(t-3)+3\delta(t-2)}{t}\right] = \frac{2}{3}e^{-3s} + \frac{3}{2}e^{-2s}$$
$$= \frac{1}{6}\left[4e^{-3s} + 9e^{-2s}\right]$$