MODULE - 2

PARTIAL DIFFERENTIATION

An equation which involves one dependent variable w.r.t two or more independent varia-bles is called Partial differential equation.

D. $u = f(x, y)$ I.D

Ex: $x \cdot \frac{\delta z}{\delta x} + y \frac{\delta z}{\delta y} = 3z$.

The derivative of u w.r. to x treating y as constant is called as the partial derivative of u w.r. to x & is denoted by $\frac{\delta u}{\delta x}$ or u_x.

$u_x = y$ const

11ly the derivative of u w.r. to y treating x as a constant is called as the Partial derivative of u w.r. to y & is denoted by $\frac{\delta u}{\delta y}$ or u_y.

$u_y - x$ constant

d

ordin dif $= \frac{dy}{dx}$

part diff $- \frac{\delta x}{\delta y}$

δ

Examples

① $y = 3x^2 + 6x + 7$

Partial diff w.r.t x.

$$\frac{dy}{dx} = 6x + 6$$

② $u = 3x^2y + 6xy^2 + 7$

Partial Diff w.r.t. x

$$\frac{\delta u}{\delta x} = 6xy + 6y^2$$

Partial Diff w.r.to y.

$$\frac{\delta u}{\delta y} = 3x^2 + 12xy$$

③ $y = e^{4x+3}$

Partial Diff w.r.t. x

$$\frac{dy}{dx} = e^{4x+3} (4)$$

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$u = e^{4x+3y}$

Diff w.r.t x

$\frac{\partial u}{\partial x} = e^{4x+3y}(4+0)$

$= 4e^{4x+3y}$

Diff w.r.t y

$\frac{\partial u}{\partial y} = e^{4x+3y}(3+0)$

$= 3e^{4x+3y}$

$y = \sin 5x$

Diff w.r.t x.

$\frac{dy}{dx} = 5.\cos 5x.$

$u = \sin(xy)$

$\frac{\partial u}{\partial x} = \cos(xy)\frac{\partial}{\partial x}(xy) \Rightarrow y.\cos(xy)$

$\frac{\partial u}{\partial y} = \cos(xy)\frac{\partial}{\partial y}(xy) \Rightarrow x.\cos(xy)$

$y = \tan^{-1}(2/x)$

$\frac{dy}{dx} = \frac{1}{1+(2/x)^2}\frac{d}{dx}(2/x) = \frac{-2}{x^2+4}$

$\frac{1}{\frac{x^2+4}{x^2}}$ $\left[\frac{x(0)-2(1)}{x^2}\right]$

$-2/x^2 \times \frac{x^2}{x^2+4}$

$u = \tan^{-1}(y/x)$

$\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2}\partial/\partial x(y/x) = \frac{-y}{x^2+y^2}$

$\frac{x(0)-y(1)}{x^2} \times \frac{x^2}{x^2+y}$

$-y/x^2+y$

$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2}\partial/\partial y(y/x) = \frac{x}{x^2+y^2}$

$\frac{d}{dx}[f(y)] = f'(y)\, dy/dx$

If r is a func of x & y

$\partial/\partial x[f(r)] = f'(r)\frac{\partial r}{\partial x}$

$\partial/\partial y[f(r)] = f'(r)\,\partial r/\partial y$

Higher order partial derivatives

$$u = f(x, y)$$

First order partial derivatives

$$u_x = \frac{\delta u}{\delta x} \qquad u_y = \frac{\delta u}{\delta y}$$

Second order partial derivatives

$$u_{xx} = \frac{\delta}{\delta x}\left[\frac{\delta u}{\delta x}\right] = \frac{\delta^2 u}{\delta x^2} \qquad u_{yy} = \frac{\delta}{\delta y}\left[\frac{\delta u}{\delta y}\right] = \frac{\delta^2 u}{\delta y^2}$$

mixed partial derivatives

$$u_{yx} = \frac{\delta}{\delta y}\left[\frac{\delta u}{\delta x}\right] = \frac{\delta^2 u}{\delta y . \delta x} \qquad u_{xy} = \frac{\delta}{\delta x}\left[\frac{\delta u}{\delta y}\right] = \frac{\delta^2 u}{\delta x . \delta y}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) \qquad y\left(\frac{-1}{x^2}\right)$$

$$\rightarrow \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{d}{dx}\left(\frac{y}{x}\right) \Rightarrow \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{-y}{x^2}$$

$$\frac{1}{\frac{x^2+y^2}{\cancel{x^2}}} \times \frac{-y}{\cancel{x^2}} = -y/x^2+y^2$$

$$\frac{1}{1 + \frac{y^2}{x^2}} \frac{d}{dy}\left(\frac{y}{x}\right) \Rightarrow \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}} \times \frac{1}{\cancel{x}} \times 1 = \frac{x}{x^2+y^2}$$

Note: In mixed order Partial derivatives $u_{yx} = u_{xy}$.

Problems :-

1] If $u = x^3 - 3xy^2 + x + e^x \cos y + 1$. S.T. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Soln: We have $u = x^3 - 3xy^2 + x + e^x \cos y + 1$

Partial differentiating w.r.t. x.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1 + e^x \cos y + 0$$

Again differentiating this w.r.t x partially

$$\frac{\partial^2 u}{\partial x^2} = 6x + e^x \cos y$$

Partial ~~diff~~ w.r.t y. x - constant

$$\frac{\partial u}{\partial y} = -6xy + e^x(-\sin y)$$

$x\cancel{y} = x$

$x\cancel{0} = 0$

Aggain diff w.r.t y partially.

$$\frac{\partial^2 u}{\partial y^2} = -6x - e^x \cos y$$

Now. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$6x + e^x \cos y + [-6x - e^x \cos y] = 0$$

$$\cancel{6x} + \cancel{e^x \cos y} - \cancel{6x} - \cancel{e^x \cos y} = 0$$

$$0 = 0$$

2] If $u = \log\left[\frac{x^2+y^2}{x+y}\right]$ S.T $xu_x + yu_y = 1$

Soln: We have $u = \log\left[\frac{x^2+y^2}{x+y}\right] \Rightarrow u = \log(x^2+y^2) - \log(x+y)$

partially diff w.r.t x $\frac{\delta u}{\delta x} = \frac{1}{x^2+y^2}\frac{\delta}{\delta x}(x^2+y^2) - \frac{1}{x+y}\frac{\delta}{\delta x}(x+y)$

$$\frac{\delta u}{\delta x} = \frac{1}{x^2+y^2}(2x) - \frac{1}{x+y}(1)$$

partially diff w.r.t y.

$$u_y = \frac{\delta u}{\delta y} = \frac{1}{x^2+y^2}\cdot 2y - \frac{1}{x+y}\cdot 1$$

Now. $x u_x + y u_y$

$$= x\left[\frac{2x}{x^2+y^2} - \frac{1}{x+y}\right] + y\left[\frac{2y}{x^2+y^2} - \frac{1}{x+y}\right]$$

$$= \frac{2x^2}{x^2+y^2} - \frac{x}{x+y} + \frac{2y^2}{x^2+y^2} - \frac{y}{x+y}$$

$$= \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2} - \frac{x}{x+y} - \frac{y}{x+y}$$

$$= \frac{2\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} - \frac{\cancel{(x+y)}}{\cancel{x+y}}$$

$$= 2 - 1$$

$$= 1$$

3) If $u = e^{ax-by}\sin(ax+by)$. S.T $b\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y} = 2abu$

Soln: we have $u = e^{ax-by}\sin(ax+by)$

partial diff w.r.t x

$$\frac{\partial u}{\partial x} = e^{ax-by}\cos(ax+by)\frac{\delta}{\delta x}(ax+by) + e^{ax-by}\frac{\delta}{\delta x}(ax-by)\cdot\sin(ax+by)$$

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$$\frac{\partial u}{\partial x} = e^{ax-by}\cos(ax+by).a + e^{ax-by}.a\sin(ax+by)$$

i.e. $$\frac{\partial u}{\partial x} = a e^{ax-by}\cos(ax+by) + au \quad \text{--- (1)}$$

partially diff w.r.t y

$$\frac{\partial u}{\partial y} = e^{ax-by}\cos(ax+by)\frac{\partial}{\partial x}(ax+by) + e^{ax-by}\frac{\partial}{\partial}(ax-by)\sin(ax+by)$$

$$\frac{\partial u}{\partial y} = e^{ax-by}\cos(ax+by).b + e^{ax-by}.(-b).\sin(ax+by)$$

$$\frac{\partial u}{\partial y} = b e^{ax-by}\cos(ax+by) - b e^{ax-by}\sin(ax+by)$$

$$\frac{\partial u}{\partial y} = b e^{ax-by}\cos(ax+by) - bu \quad \text{--- (2)}$$

Now. $b\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y}$

by using (1) & (2) becomes.

$$= b[a e^{ax-by}\cos(ax+by) + au] - a[b e^{ax-by}\cos(ax+by) - bu]$$

$$= ab e^{ax-by}\cancel{\cos(ax+by)} + abu - ab e^{ax-by}\cancel{\cos(ax+by)} + abu$$

$$= \underline{\underline{2abu}}$$

47. If $z = \sinh^{-1}(x/y)$. S.T $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Soln: We have $z = \sinh^{-1}(x/y)$

partially diff w.r.t x

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1+(x/y)^2}}\frac{\partial}{\partial x}(x/y)$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1+(x^2/y^2)}} \cdot \frac{1}{y}$$

$$= \frac{1}{\sqrt{\frac{y^2+x^2}{y^2}}} \cdot \frac{1}{y} \Rightarrow \frac{\partial z}{\partial x} = \frac{y}{\sqrt{y^2+x^2}} \cdot \frac{1}{y}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{y^2+x^2}}$$

Partially diff w.r.t y.

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1+(x/y)^2}} \cdot \frac{\partial}{\partial y}(x/y).$$

$$= \frac{1}{\sqrt{\frac{y^2+x^2}{y^2}}} \left(x \cdot \frac{-1}{y^2} \right)$$

$$= \frac{-x}{\sqrt{y^2+x^2}} \cdot \frac{1}{y^2} \times y$$

$$\frac{\partial z}{\partial y} = \frac{-x}{y\sqrt{y^2+x^2}}$$

Now, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

$$= x\left[\frac{1}{\sqrt{y^2+x^2}}\right] + y\left[\frac{-x}{y\sqrt{y^2+x^2}}\right]$$

$$= \frac{x}{\sqrt{y^2+x^2}} - \frac{x}{\sqrt{y^2+x^2}}$$

$$= 0$$

5] If $u = \tan^{-1}(y/x)$ verify that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

Soln:. We have $u = \tan^{-1}(y/x)$

Partially diff w.r.t x

$$\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2} \frac{\partial}{\partial x}(y/x)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}} (y \cdot {}^{-1}/x^2) \Rightarrow = \frac{-y}{x^2+y^2} \cdot \frac{1}{x^2} \times x^2$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2}$$

Diff w.r.t y partially.

$$\frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2} \frac{\partial}{\partial y}(y/x)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}} (1/x \cdot 1) \Rightarrow = \frac{1}{x^2+y^2} \cdot \frac{1}{x} \times x^2$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$$

Now, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]$

$$= \frac{\partial}{\partial y}\left[\frac{-y}{x^2+y^2}\right]$$

by applying quotient rule

$$= \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Also $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]$

$$= \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right]$$

$$= \frac{(x^2+y^2)(1) - (x).(2x)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} \Rightarrow = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x . \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x . \partial y}$$

6]. If $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$. Show that

$$\frac{\partial^2 z}{\partial x . \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

Soln :- we have . $z = x^2 \tan^{-1}(y/x) - y^2 \underline{\tan^{-1}(x/y)}$

product rule

partial diff w.r.t y.

$$\frac{\partial z}{\partial y} = x^2 \frac{1}{1+(y/x)^2} \cdot 1/x - \left[y^2 \frac{1}{1+(x/y)^2}(x. -1/y^2) + 2y \tan^{-1}(x/y)\right]$$

$$= \frac{x^2}{\frac{x^2+y^2}{x^2}} \cdot 1/x - \left[\frac{-xy^2}{\frac{y^2+x^2}{y^2}} \cdot 1/y^2 + 2y \tan^{-1}(x/y)\right]$$

$$= \frac{x^2}{x^2+y^2} \cdot 1/\cancel{x} \cdot x^2 + \frac{xy^2}{x^2+y^2} \, 1/\cancel{y^2} \cdot \cancel{y^2} - 2y \tan^{-1}(x/y)$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \tan^{-1}(x/y)$$

$$\frac{\partial z}{\partial y} = \frac{x^3+xy^2}{x^2+y^2} - 2y \tan^{-1}(x/y)$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y \tan^{-1}(x/y)$$

$$\therefore \frac{\partial z}{\partial y} = x - 2y \tan^{-1}(x/y)$$

Again partially diff w.r.t x.

$$\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = 1 - 2y\left[\frac{1}{1+(x/y)^2}\,(1/y)\right]$$

$$= 1 - 2y\left[\frac{1}{\frac{y^2+x^2}{y^2}} \cdot 1/y\right]$$

$$= 1 - 2y\left[\frac{1}{x^2+y^2} \cdot 1/y \cdot y^2\right]$$

$$= 1 - 2y\left[\frac{y}{x^2+y^2}\right]$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$\frac{\delta^2 z}{\delta x \delta y} = \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x}$$

7] If $u = \sin^{-1}(y/x)$. S.T. $u_{xy} = u_{yx}$

Soln :- we have $u = \sin^{-1}(y/x)$

Partially diff w.r.t. y

$$\frac{\delta u}{\delta y} = \frac{1}{\sqrt{1-(y/x)^2}} \cdot 1/x \,(1)$$

$$= \frac{1}{\sqrt{\frac{x^2-y^2}{x^2}}} \cdot 1/x \quad = \frac{1}{\sqrt{x^2-y^2}} \cdot 1/x \cdot x$$

$$\frac{\delta u}{\delta y} = \frac{1}{\sqrt{x^2-y^2}}$$

Partially diff w.r.t. x

$$\frac{\delta u}{\delta x} = \frac{1}{\sqrt{1-(y/x)^2}} \cdot y \cdot -1/x^2$$

$$= \frac{-y}{\sqrt{\frac{x^2-y^2}{x^2}}} \cdot 1/x^2 \implies \frac{\delta u}{\delta x} = \frac{-y}{\sqrt{x^2-y^2}} \cdot 1/x^2 \cdot x$$

$$\frac{\delta u}{\delta x} = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$u_{xy} = \frac{\delta^2 u}{\delta x \delta y} = \frac{\delta}{\delta x}\left[\frac{\delta u}{\delta y}\right]$$

$$= \frac{\delta}{\delta x}\left[\frac{1}{\sqrt{x^2+y^2}}\right] \implies u_{xy} = \frac{\delta}{\delta x}\left[(x^2+y^2)^{-1/2}\right]$$

$$u_{xy} = -1/2\,(x^2+y^2)^{-1/2-1} \cdot 2x \rightarrow -x \cdot (x^2-y^2)^{-3/2}$$

$$u_{xy} = \frac{-x}{(x^2-y^2)^{3/2}}$$

$$u_{yx} = \frac{\delta^2 u}{\delta y \delta x} = \frac{\delta}{\delta y}\left[\frac{\delta u}{\delta x}\right] = \frac{\delta}{\delta y}\left[\frac{-y}{x\sqrt{x^2-y^2}}\right]$$

$$u_{yx} = \frac{-1}{x}\left[\sqrt{x^2-y^2}\,.\,1 - y\right.$$

8]. If $u = x^y$ S.T. $u_{xy} = u_{yx}$.

Soln :- we have $u = x^y$ $\qquad \frac{d}{dx}(x^n) = nx^{n-1}$

partially diff w.r.t x

$$\frac{\delta u}{\delta x} = y x^{y-1}$$

partially diff w.r.t. y

$$\frac{\delta u}{\delta y} = x^y \log x \qquad \frac{d}{dx}(a^x) = a^x \log a$$

$$u_{xy} = \frac{\delta^2 u}{\delta x \delta y} \Rightarrow \frac{\delta}{\delta x}\left[\frac{\delta u}{\delta y}\right] = \frac{\delta}{\delta x}\left[x^y \log x\right)$$

$$u_{xy} = x^y \frac{1}{x} + y x^{y-1} \log x$$

$$u_{xy} = x^{y-1} + y x^{y-1} \log x$$

$$u_{xy} = x^{y-1}(1 + y \log x) \quad — (1)$$

$$u_{yx} = \frac{\delta^2 u}{\delta y \delta x} = \frac{\delta}{\delta y}\left[\frac{\delta}{\delta x}\right]$$

$$= \frac{\delta}{\delta y}\left[y x^{y-1}\right]$$

$$u_{yx} = y x^{y-1} \log x + x^{y-1}.1$$

$$u_{yx} = x y^{-1}(y \log x + 1) \quad — (2)$$

From (1) & (2) $u_{xy} = u_{yx}$

9] If $u = e^x[x\cos y - y\sin y]$. S.T $u_{xy} = u_{yx}$

Soln: We have $u = e^x[x\cos y - y\sin y]$

partially diff w.r.t x

$$\frac{\partial u}{\partial x} = u_x = e^x \cdot 1 \cdot \cos y + e^x[x\cos y - y\sin y]$$

partially diff w.r.t y

$$\frac{\partial u}{\partial y} = u_y = e^x[-x\sin y - y\cos y - 1 \cdot \sin y]$$

$$u_y = -e^x[x\sin y + y\cos y + \sin y]$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]$$

$$u_{xy} = \frac{\partial}{\partial x}\left[-e^x[x\sin y + y\cos y + \sin y]\right]$$

$$u_{xy} = -\left[e^x \cdot 1 \cdot \sin y + e^x[x\sin y + y\cos y + \sin y]\right]$$

$$u_{xy} = -e^x[2\sin y + x\sin y + y\cos y] \cancel{+\sin y} \quad \text{— (1)}$$

$$u_{yx} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[e^x \cos y + e^x[x\cos y - y\sin y]\right]$$

$$= \frac{\partial}{\partial y}\left[e^x[\cos y + x\cos y - y\sin y]\right]$$

$$= e^x[-\sin y - x\sin y - y\cos y - (1)\sin y]$$

$$= -e^x[\sin y + x\sin y + y\cos y + \sin y]$$

$$u_{yx} = -e^x[2\sin y + x\sin y + y\cos y] \quad \text{— (2)}$$

From (1) & (2)

$$u_{xy} = u_{yx}$$

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10] If $u = e^{ax+by} f(ax-by)$. S.T $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$

Soln :- We have. $u = e^{ax+by} f(ax-by)$

partially diff w.r.t x.

$$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)(a) + f(ax-by)e^{ax+by}\cdot(a)$$

$$\frac{\partial u}{\partial x} = a e^{ax+by} f'(ax-by) + a\, e^{ax+by} f(ax-by)$$

$$\frac{\partial u}{\partial x} = a e^{ax+by} f'(ax-by) + au.$$

partially diff w.r.t. y

$$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)e^{ax+by}(b)$$

$$\frac{\partial u}{\partial y} = -b\, e^{ax+by} f'(ax-by) + b\, e^{ax+by} f(ax-by)$$

$$\frac{\partial u}{\partial y} = -b e^{ax+by} f'(ax-by) + bu$$

Now, $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2aba$

$$b\left[a e^{ax+by} f'(ax-by) + au\right] + a\left[-b e^{ax+by} f'(ax-by) + bu\right]$$

$$ab e^{ax+by} f'(ax-by) + abu - ab e^{ax+by} f'(ax-by) + abu$$

$$abu + abu = 2abu$$

$$2abu = 2abu$$

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11] If $u = f(x+ct) + g(x-ct)$. S.T $u_{tt} = c^2 u_{xx}$

Soln: We have $u = f(x+ct) + g(x-ct)$

Partially diff w.r.t t.

$u_t = f'(x+ct) \cdot c + g'(x-ct)(-c)$

Again partially diff w.r.t t.

$u_{tt} = f''(x+ct)\, c^2 + g''(x-ct) \cdot c^2$

Partially diff w.r.t. x.

$u_x = f'(x+ct)(1) + g'(x-ct)(1)$

Again partially diff w.r.t. x.

$u_{xx} = f''(x+ct)(1) + g''(x-ct)(1)$

Now,

$u_{tt} = f''(x+ct)\, c^2 + g''(x-ct)\, c^2$

$u_{tt} = c^2 [f''(x+ct) + g''(x-ct)]$

$u_{tt} = c^2 u_{xx}$

12] If $u = \log\sqrt{x^2+y^2+z^2}$. S.T $(x^2+y^2+z^2)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] =$

Soln: We have $u = \log\sqrt{x^2+y^2+z^2}$

$u = \log((x^2+y^2+z^2)^{1/2})$ $\qquad \log(x^p) = p\log(x)$

$u = \frac{1}{2}\log(x^2+y^2+z^2)$

The given u is a symmetric func of $x.y.z$

Partially diff w.r.t x.

$$\frac{\delta u}{\delta x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2+z^2} \cdot 2x$$

$$\frac{\delta u}{\delta x} = \frac{x}{x^2+y^2+z^2}$$

Again partial diff w.r.t x

$$\frac{\delta^2 u}{\delta x^2} = \frac{(x^2+y^2+z^2)(1) - x(2x)}{(x^2+y^2+z^2)^2}$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} \quad \text{---} (1)$$

similarly, $\frac{\delta^2 u}{\delta y^2} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}$ --- (2)

$$\frac{\delta^2 u}{\delta z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \quad \text{---} (3)$$

Adding (1), (2) & (3) we get

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} + \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{\cancel{y^2}+\cancel{z^2}-\cancel{x^2}+\cancel{x^2}+z^2-\cancel{y^2}+x^2+y^2-\cancel{z^2}}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2} = \frac{1}{x^2+y^2+z^2}$$

$$(x^2+y^2+z^2)\left(\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2}\right) = 1$$

13]. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ then S.T $\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2} = 0$

Soln: We have $u = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$ is a symmetric fun of x, y z.

Partially diff w.r.t. x. $\quad x^n = nx^{n-1}$

$$\frac{\delta u}{\delta x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$

$$\frac{\delta u}{\delta x} = -(x^2+y^2+z^2)^{-3/2} \cdot x.$$

Again partial diff w.r.t. x

$$\frac{\delta^2 u}{\delta x^2} = -\left[(x^2+y^2+z^2)^{-3/2} \cdot 1 + x\left[-3/2 (x^2+y^2+z^2)^{-5/2} \cdot 2x\right]\right]$$

$$= -\left[(x^2+y^2+z^2)^{-3/2} - (3x^4 + 3x^2y^2 - 3x^2z^2)\right]^{-5/2}$$

$$= -\left[(x^2+y^2+z^2)^{-3/2} - 3x^2\left[x^2+y^2+z^2\right]^{-5/2}\right]$$

$$\frac{\delta^2 u}{\delta x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (1)}$$

Similarly, $\frac{\delta^2 u}{\delta y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$ --- (2)

$$\frac{\delta^2 u}{\delta z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (3)}$$

Adding the results (1), (2) & (3) we have

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$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = 0$$

$$\Rightarrow 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2} -$$

$$(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} = 0$$

$$\Rightarrow 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2} = 0$$

$$3(x^2+y^2+z^2)^{-5/2+1} - 3(x^2+y^2+z^2)^{-3/2} = 0$$

$$3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$$

$$0 = 0$$

14] If $z(x+y) = x^2+y^2$. S.T $\left[\frac{\delta z}{\delta x} - \frac{\delta z}{\delta y}\right]^2 = 4\left[1 - \frac{\delta z}{\delta x} - \frac{\delta z}{\delta y}\right]$

Soln :- we have $z(x+y) = x^2+y^2$

$$z = \frac{x^2+y^2}{x+y}$$ is a Symmetric func of x, y

partial diff w.r.t x

$$\frac{\delta z}{\delta x} = \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$\frac{\delta z}{\delta x} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Similarly. $$\frac{\delta z}{\delta y} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

NOW, $\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}$

$$= \frac{x^2+2xy-y^2-y^2-2xy+x^2}{(x+y)^2}$$

$$\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = \frac{2x^2-2y^2}{(x+y)^2}$$

$$= \frac{2(x^2-y^2)}{(x+y)^2} = \frac{2(x-y)\cancel{(x+y)}}{(x+y)^{\cancel{2}}}$$

$$= \frac{2(x-y)}{(x+y)}$$

Square on b-s.

$$\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2 = 4\,\frac{(x-y)^2}{(x+y)^2} \quad \text{— (1)}$$

NOW, $4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = 4\left[1 - \left(\frac{x^2+2xy-y^2}{(x+y)^2}\right) - \left(\frac{y^2+2xy-x^2}{(x+y)^2}\right)\right]$

$$= 4\left[\frac{(x+y)^2 - \cancel{x^2} - 2xy + \cancel{y^2} - \cancel{y^2} - 2xy + \cancel{x^2}}{(x+y)^2}\right]$$

$$= 4\left[\frac{(x+y)^2 - 4xy}{(x+y)^2}\right] \Rightarrow = 4\left[\frac{x^2+y^2+2xy-4xy}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2+y^2-2xy}{(x+y)^2}\right] \Rightarrow = 4\left[\frac{(x-y)^2}{(x+y)^2}\right] \quad \text{— (2)}$$

Tus from eq^{ns} (1) & (2) $\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$

15] If ~~a~~ $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$ Show that

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{\delta u}{\delta t}$$

Soln: We have $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$

partially diff w.r.t x

$$\frac{\delta u}{\delta x} = e^{-2\pi^2 t} (\pi \cos \pi x) \sin \pi y$$

Again partially diff w.r.t x.

$$\frac{\delta^2 u}{\delta x^2} = e^{-2\pi^2 t} (-\pi^2 \sin \pi x) . \sin \pi y$$

$$\frac{\delta^2 u}{\delta x^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi x . \sin \pi y$$

$$\frac{\delta^2 u}{\delta x^2} = -\pi^2 u$$

partially diff w.r.t y

$$\frac{\delta u}{\delta y} = e^{-2\pi^2 t} \sin \pi x . (\pi \cos \pi y)$$

Again partially diff w.r.t y

$$\frac{\delta^2 u}{\delta y^2} = e^{-2\pi^2 t} \sin \pi x (-\pi^2 \sin \pi y)$$

$$\frac{\delta^2 u}{\delta y^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi x \sin \pi y$$

$$\frac{\delta^2 u}{\delta y^2} = -\pi^2 u$$

Thus LHS. $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}$

$= -\pi^2 u - \pi^2 u$

$= -2\pi^2 u$ —— (1)

Also, partially diff w.r.t t.

$$\frac{\delta u}{\delta t} = e^{-2\pi^2 t}(-2\pi^2)\sin\pi x \sin\pi y$$

$$\frac{\delta u}{\delta t} = -2\pi^2 e^{-2\pi^2 t}\sin\pi x . \sin\pi y$$

Thus $\underset{RHS}{\frac{\delta u}{\delta t}} = -2\pi^2 u \quad —(2)$

$\therefore$ from (1) & (2) $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{\delta u}{\delta t}$

16] If $u = \log(x^3+y^3+z^3-3xyz)$ then prove that $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = \frac{3}{x+y+z}$ and hence show that

$$\left(\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Soln: $u = \log(x^3+y^3+z^3-3xyz)$ is a symmetric func

partially diff w.r.t x.

$$\frac{\delta u}{\delta x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz) \quad —(1)$$

partially diff w.r.t y

$$\frac{\delta u}{\delta y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz) \quad —(2)$$

partially diff w.r.t z

$$\frac{\delta u}{\delta z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy) \quad —(3)$$

Now. $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z}$

$= \cancel{\frac{3x^2-3yz}{x^3}}$

$$= \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} + \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz} + \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3x^2-3yz+3y^2-3zx+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

Recalling a standard elementary result

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = \frac{3}{x+y+z}.$$

Further, $\left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right]^2 u$

$$= \left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right]\left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right] u$$

$$= \left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right]\left[\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z}\right]$$

$$= \left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right]\left[\frac{3}{x+y+z}\right]$$

$$= \frac{\delta}{\delta x}\left[\frac{3}{x+y+z}\right] + \frac{\delta}{\delta y}\left[\frac{3}{x+y+z}\right] + \frac{\delta}{\delta z}\left[\frac{3}{x+y+z}\right]$$

$$= \frac{(x+y+z)(0)-3(1)}{(x+y+z)^2} + \frac{(x+y+z)(0)-3(1)}{(x+y+z)^2} + \frac{(x+y+z)(0)-3(1)}{(x+y+z)^2}$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2}$$

$$= \frac{-3-3-3}{(x+y+z)^2}$$

$$\left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

JACOBIANS

Let "u" and "v" be the function of two inde-pendent variables 'x' and 'y'. The Jacobian "J" of the functions "u" and "v" w.r.t "x" & "y" is symbolically represented and is defined as

$$J\left(\frac{u.v}{x.y}\right) = \frac{\delta(u.v)}{\delta(x.y)} = \begin{vmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} \end{vmatrix}$$

If "u". "v" and "w" are function of "x", "y" and "z" can be written as.

$$J\left(\frac{u.v.w}{x.y.z}\right) = \frac{\delta(u.v.w)}{\delta(x.y.z)} = \begin{vmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} & \frac{\delta u}{\delta z} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} & \frac{\delta v}{\delta z} \\ \frac{\delta w}{\delta x} & \frac{\delta w}{\delta y} & \frac{\delta w}{\delta z} \end{vmatrix}$$

problems.

(1) If $u = x^2 - 2y$. and $v = x+y$. find $J\left(\frac{u.v}{x.y}\right)$

Soln :- $J\left(\frac{u.v}{x.y}\right) = \begin{vmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} \end{vmatrix}$

$u = x^2 - 2y$

w.r.t x

$\frac{\delta u}{\delta x} = 2x$

w.r.t. y

$\frac{\delta u}{\delta y} = -2$

$v = x+y$

w.r.t x

$\delta v/\delta x = 1$

w.r.t y

$\delta v/\delta y = 1$

$$J\left(\frac{u \cdot v}{x \cdot y}\right) = \begin{vmatrix} 2x & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 2x(1) - (-2)(1)$$

$$= 2x + 2$$

$$J\left(\frac{u \cdot v}{x \cdot y}\right) = 2(x+1)$$

2] $u = x^2 - 2y$. $v = x + y + z$. $w = x - 2y + 3z$. find $\frac{\partial(u \cdot v \cdot w)}{\partial(x \cdot y \cdot z)}$.

Solu :-

$$J\left(\frac{u \cdot v \cdot w}{x \cdot y \cdot z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u = x^2 - 2y$	$v = x + y + z$	$w = x - 2y + 3z$
$u_x = 2x$	$v_x = 1$	$w_x = 1$
$u_y = -2$	$v_y = 1$	$w_y = -2$
$u_z = 0$	$v_z = 1$	$w_z = 3$

$$J\left(\frac{u \cdot v \cdot w}{x \cdot y \cdot z}\right) = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3 - (-2)) - (-2)(3 - 1) + 0(-2 - 1)$$

$$= 2x(3 + 2) + 2(+2) + 0$$

$$= 10x + 4$$

3] If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$ and $w = \frac{xy}{z}$ find $\frac{\partial(u.v.w)}{\partial(x.y.z)}$

Soln: $J\left(\frac{u.v.w}{x.y.z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

$u = \frac{yz}{x}$ $\quad v = \frac{zx}{y}$ $\quad w = \frac{xy}{z}$

$u_x = \frac{-yz}{x^2}$ $\quad v_x = \frac{z}{y}$ $\quad w_x = \frac{y}{z}$

$u_y = \frac{z}{x}$ $\quad v_y = \frac{-zx}{y^2}$ $\quad w_y = \frac{x}{z}$

$u_z = \frac{y}{x}$ $\quad v_z = \frac{x}{y}$ $\quad w_z = \frac{-xy}{z^2}$

$$J\left(\frac{u.v.w}{x.y.z}\right) = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ z/y & -zx/y^2 & x/y \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left[\left(\frac{-zx}{y^2}\right)\left(\frac{-xy}{z^2}\right) - \left(x/y\right)\left(x/z\right)\right] - \frac{z}{x}\left[\left(\frac{z}{y}\right)\left(\frac{-xy}{z^2}\right) - (x/y)(y/z)\right] + \frac{y}{x}\left[\left(z/y\right)\left(x/z\right) - \left(-zx/y^2\right)\left(y/z\right)\right]$$

$$= \frac{-yz}{x^2}\left[\cancel{\frac{x^2}{yz}} - \cancel{\frac{x^2}{yz}}\right] - \frac{z}{x}\left[-\frac{x}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$= \frac{-z}{x}\left[-\frac{2x}{z}\right] + \frac{y}{x}\left[2\frac{x}{y}\right]$$

$$= \frac{2zx}{zx} + \frac{2xy}{xy}$$

$$= 2 + 2$$

$$J\left(\frac{u.v.w}{x.y.z}\right) = 4$$

4] If $u = x + 3y^2 - z^3$. $v = 4x^2yz$. $w = 2z^2 - xy$ then find $J\left(\frac{u.v.w}{x.y.z}\right)$ at $(1, -1, 0)$

Solu:- $J\left(\frac{u.v.w}{x.y.z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

$u = x + 3y^2 - z^3$	$v = 4x^2yz$	$w = 2z^2 - xy$
$u_x = 1$	$v_x = 8xyz$	$w_x = -y$
$u_y = 6y$	$v_y = 4x^2z$	$w_y = -x$
$u_z = -3z^2$	$v_z = 4x^2y$	$w_z = 4z$

$$J\left(\frac{u.v.w}{x.y.z}\right) = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$= 1\left[(4x^2z)(4z) - (4x^2y)(-x)\right] - 6y\left[(8xyz)(4z) - (4x^2y)(-y)\right] + (-3z^2)\left[(8xyz)(-x) - (4x^2z)(-y)\right]$$

$$= 1\left[16x^2z^2 + 4x^3y\right] - 6y\left[32xyz^2 + 4x^2y^2\right] - 3z^2\left[-8x^2yz + 4x^2yz\right]$$

$= 16x^2z^2 + 4x^3y - 192xy^2z^2 - 24x^2y^3 + 24x^2yz^3 - 12x^2yz^3.$

at $(1, -1, 0)$

$= 16(1)^2(0)^2 + 4(1)^3(-1) - 192(1)(-1)^2(0)^2 - 24(1)^2(-1)^3 + 24(1)^2(-1)(0)^3 - 12(1)^2(-1)(0)^3$

$= -4 + 24$

$J(1, -1, 0) = 20$

5] If $u = \dfrac{x+y}{1-xy}$. $v = \tan^{-1}x + \tan^{-1}y$ find $\dfrac{\partial(u, v)}{\partial(x, y)}$

Solu :- $J\left(\dfrac{u, v}{x, y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$u = \dfrac{x+y}{1-xy}$

$u_x = \dfrac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2}$

$= \dfrac{1 - xy + xy + y^2}{(1-xy)^2}$

$u_x = \dfrac{1+y^2}{(1-xy)^2}$

$u_y = \dfrac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2}$

$= \dfrac{1 - xy + x^2 + xy}{(1-xy)^2}$

$= \dfrac{1+x^2}{(1-xy)^2}$

$v = \tan^{-1}x + \tan^{-1}y$

$v_x = \dfrac{1}{1+x^2}$

$v_y = \dfrac{1}{1+y^2}$

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$$J\left(\frac{u.v}{x.y}\right) = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1+y^2}{(1-xy)^2} \times \frac{1}{1+y^2} - \frac{1+x^2}{(1+xy)^2} \times \frac{1}{1+x^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$J\left(\frac{u.v}{x.y}\right) = 0$$

6] If $x+y+z=u$, $y+z=v$ and $z=uvw$ show that $\frac{\delta(x.y.z)}{\delta(u.v.w)}$

Soln :- $\frac{\delta(x.y.z)}{\delta(u.v.w)} = \begin{vmatrix} \delta x/\delta u & \delta x/\delta v & \delta x/\delta w \\ \delta y/\delta u & \delta y/\delta v & \delta y/\delta w \\ \delta z/\delta u & \delta z/\delta v & \delta z/\delta w \end{vmatrix}$

consider. $x+y+z=u$ — (1)

$y+z=v$ — (2)

$z=uvw$ — (3)

using (2) in (1) we have

$x+v=u \Rightarrow \therefore x=u-v$

Also by using (3) in (2) we have

$y+uvw=v$. $\therefore y=v-uvw$.

$\therefore x=u-v$, $y=v-uvw$

$z=uvw$

$x = u - v \qquad y = v - uvw \qquad z = uvw$

$x_u = 1 \qquad y_u = -vw \qquad z_u = vw$

$x_v = -1 \qquad y_v = 1 - uw \qquad z_v = uw$

$x_w = 0 \qquad y_w = -uv \qquad z_w = uv$

$$\therefore \frac{\delta(x.y.z)}{\delta(u.v.w)} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= 1\left[(1-uw)(uv) + (uv)(uw)\right] + 1\left[(-vw)(uv) + (uv)(vw)\right] + 0\left[(-vw)(uw) - (1-uw)(vw)\right].$$

$$= uv - \cancel{u^2vw} + \cancel{u^2vw} - \cancel{uv^2w} + \cancel{uv^2w}$$

$$= uv$$

$$\therefore \frac{\delta(x.y.z)}{\delta(u.v.w)} = uv$$

7] If $x = r\sin\theta\cos\phi$. $y = r\sin\theta\sin\phi$. $z = r\cos\theta$.

Show that $\dfrac{\delta(x.y.z)}{\delta(r.\theta.\phi)} = r^2\sin\theta$

Soln: $J\left(\dfrac{x.y.z}{r.\theta.\phi}\right) = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} & \frac{\delta x}{\delta \phi} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} & \frac{\delta y}{\delta \phi} \\ \frac{\delta z}{\delta r} & \frac{\delta y}{\delta \theta} & \frac{\delta y}{\delta \phi} \end{vmatrix}$

$x = r\sin\theta . \cos\phi$

$\dfrac{\delta x}{\delta r} = \sin\theta . \cos\phi \qquad \dfrac{\delta x}{\delta \phi} = r\sin\theta \; -\sin\phi$

$\dfrac{\delta x}{\delta \theta} = r\cos\theta . \cos\phi$

$$y = r \sin\theta \sin\phi \qquad z = r\cos\theta$$

$$\frac{\delta y}{\delta r} = 1 \sin\theta \sin\phi \qquad \frac{\delta z}{\delta r} = \cos\theta$$

$$\frac{\delta y}{\delta \theta} = r\cos\theta \sin\phi \qquad \frac{\delta z}{\delta \theta} = r - \sin\theta$$

$$\frac{\delta y}{\delta \phi} = r \sin\theta \cos\phi \qquad \frac{\delta z}{\delta \phi} = 0$$

$$J\left(\frac{x.y.z}{r.\theta.\phi}\right) = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi[0 + r^2\sin^2\theta\cos\phi] - r\cos\theta\cos\phi[0 - r\sin\theta\cos\phi\cos\theta] - r\sin\theta\sin\phi[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\cos^2\theta\cos^2\phi\sin\theta + r^2\sin^3\theta\sin^2\phi + r^2\cos^2\theta\sin^2\phi\sin\theta$$

$$= r^2\sin^3\theta[\cos^2\phi + \sin^2\phi] + r^2\cos^2\theta\sin\theta[\cos^2\phi + \sin^2\phi]$$

$$= r^2\sin^3\theta(1) + r^2\cos^2\theta\sin\theta(1)$$

$$= r^2\sin^3\theta + r^2\cos^2\theta\sin\theta$$

$$= r^2\sin\theta(\sin^2\theta + \cos^2\theta)$$

$$= r^2\sin\theta(1)$$

$$J\left(\frac{x.y.z}{r.\theta.\phi}\right) = \underline{\underline{r^2\sin\theta}}$$

Taylor Series for function of two Variables

Taylor series for function of two variables $f(x.y)$ to the power of $(x-a)$ & $(y-b)$ about the points a & b is given by

$$f(x.y) = f(a.b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a.b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a.b) + 2(x-a)(y-b) f_{xy}\right] + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a.b) + (y-b)^3 f_{yyy}(a.b) + 3(x-a)^2 (y-b) f_{xxy} + 3(x-a)(y-b)^2 f_{xyy}\right] + \cdots$$

In particular the expansion of $f(x,y)$ in the power of x & y i.e, about the point (0,0) then we have.

$$f(x.y) = f(0.0) + \frac{1}{1!}\left[x f_x(0.0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{yy}(0,0) + y^2 f_{yy}(0.0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0.0) + y^3 f_{yyy}(0.0) + 3(x^2)(y) f_{xxy} + 3(x)(y^2) f_{xyy}\right] + \cdots$$

This is known as maclaurin series for the function of two variables about the point (0,0)

NOTE :- ① $f_{xy} = \frac{\delta^2 u}{\delta x . \delta y} = \frac{\delta}{\delta x}\left[\frac{\delta u}{\delta y}\right]$ ② $f_{xxy} = \frac{\delta}{\delta x}\left[\frac{\delta^2 u}{\delta x . \delta y}\right]$ ③ $f_{yyx} = \frac{\delta}{\delta x}\left[\frac{\delta^2 u}{\delta y^2}\right]$

Problems

1] Expand $e^x \cos y$ by Taylor's theorem about the point $(1, \pi/4)$ up to second degree.

Soln :- $f(x,y) = f(a.b) + \frac{1}{1!}\left[(x-a) f_x(a.b) + (y-b) f_y(a.b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a.b) + (y-b)^2 f_{yy}(a.b) + 2(x-a)(y-b) f_{xy}\right]$

$f(x.y) = e^x \cos y.$

$a = 1. \quad b = \pi/4.$

$$f(x,y) = f(1,\pi/4) + \frac{1}{1!}\left[(x-1) f_x(1,\pi/4) + (y-\pi/4) f_y(1,\pi/4)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\pi/4) + (y-\pi/4)^2 f_{yy}(1,\pi/4) + 2(x-1)(y-\pi/4) f_{xy}(1,\pi/4)\right] \quad \text{---} (1)$$

consider, $f(x,y) = e^x \cos y$.

$$f(1,\pi/4) = e^1 \cos(\pi/4)$$

$$f(1,\pi/4) = e \cdot 1/\sqrt{2} \quad e/\sqrt{2}$$

$$f_x(x,y) = e^x \cos y$$
$$f_x(1,\pi/4) = e^1 \cos(\pi/4)$$
$$f_x(1,\pi/4) = e \cdot 1/\sqrt{2} \quad e/\sqrt{2}$$

$$f_y(x,y) = e^x(-\sin y)$$
$$f_y(1,\pi/4) = e^1(-\sin \pi/4)$$
$$f_y(1,\pi/4) = e \; -1/\sqrt{2} \quad -e/\sqrt{2}$$

$$f_{xx}(x,y) = e^x \cos y$$
$$f_{xx}(1,\pi/4) = e^1 \cos \pi/4$$
$$f_{xx}(1,\pi/4) = e \cdot 1/\sqrt{2} \quad e/\sqrt{2}$$

$$f_{yy}(x,y) = e^x(-\cos y)$$
$$f_{yy}(1,\pi/4) = e^1(-\cos \pi/4)$$
$$f_{yy}(1,\pi/4) = e \; -1/\sqrt{2} \quad -e/\sqrt{2}$$

$\frac{\delta^2 u}{\delta x \cdot \delta y}$ $\quad f_{xy}(x,y) = e^x \cdot (-\sin y)$ $\quad \frac{\delta^2}{\delta x}\left[\frac{\delta u}{\delta y}\right]$

$$f_{xy}(1,\pi/4) = e^1(-\sin \pi/4) \quad \frac{\delta}{\delta x}\left[e^x \cdot -\sin y\right]$$
$$e^x(-\sin y)$$
$$f_{xy}(1,\pi/4) = e \cdot -1/\sqrt{2}$$

substitute all values in eq (1).

$$f(x,y) = \frac{e}{\sqrt{2}} + \frac{1}{1}\left[(x-1)\frac{e}{\sqrt{2}} + (y-\pi/4) \frac{-e}{\sqrt{2}}\right] + \frac{1}{2}\left[(x-1)^2 \frac{e}{\sqrt{2}} + (y-\pi/4)^2 \; -e/\sqrt{2} + 2(x-1)(y-\pi/4) \; -e/\sqrt{2}\right]$$

2] Expand $xy^2 + \cos(xy)$ about the point $(1, \pi/2)$ up to second degree.

Soln:- $f(x,y) = xy^2 + \cos(xy)$

$a = x = 1 \quad . \quad y = b = \pi/2$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)\right] \quad — (1)$$

let $f(x,y) = xy^2 + \cos(xy)$

$f(1, \pi/2) = 1 (\pi/2)^2 + \cos(1 . \pi/2)$

$= \frac{\pi^2}{4} + 0$

$f(1, \pi/2) = \frac{\pi^2}{4}$

$f_x(x,y) = y^2 - \sin(xy) y (1)$

$f_x(1, \pi/2) = (\pi/2)^2 - \sin(1 . \pi/2) \pi/2 (1)$

$f_x(1, \pi/2) = \frac{\pi^2}{4} - \frac{\pi}{2}$

$f_{xx}(x,y) = -y\cos(xy) . y$

$f_{xx}(1, \pi/2) = -\pi/2 . \cos(1 . \pi/2) \pi/2$

$f_{xx}(1, \pi/2) = 0$

consider. $f(x,y) = xy^2 + \cos(xy)$

$f_y(x,y) = 2xy - \sin xy . x (1)$

$f_y(1, \pi/2) = 2(1)(\pi/2) - \sin(1 . \pi/2) . (1)$

$f_y(1, \pi/2) = \pi - 1$

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$f_{yy}(x.y) = 2x - x\cos(xy) \cdot x.$

$f_{yy}(1.\pi/2) = 2(1) - 1\cos(1.\pi/2) \cdot 1.$

$f_{yy}(1.\pi/2) = 2$

$f(x.y) = xy^2 + \cos(xy)$

$\frac{\delta^2 u}{\delta x \delta y} = \frac{\delta}{\delta x}\left[\frac{\delta u}{\delta y}\right]$

$f_{xy}(x.y) = 2y(1) - [\sin(xy)]$

$\frac{\delta}{\delta x}[2yx - \sin(xy) \cdot x]$

$f_{xy}(x.y) = \frac{\delta^2 u}{\delta x \delta y}$

$f_{xy}(x.y) = \frac{\delta}{\delta x}\left[\frac{\delta u}{\delta y}\right]$

$= \frac{\delta}{\delta x}[2xy - \sin(xy) \cdot x(1)]$

$= 2y - [\sin(xy)(1) + x\cos(xy) \cdot y]$

$f_{xy}(1.\pi/2) = 2(\pi/2) - [\sin(\pi/2) \cdot + 1\cos\pi/2 \cdot \pi/2]$

$f_{xy}(1.\pi/2) = \pi - 1$

Substitute all the values in eq^n (5)

$xy^2 + \cos(xy) = f(1,\pi/2) + \frac{1}{1}\left[(x-1)\left(\frac{\pi^2}{4} - \pi/2\right) + (y - \pi/2)\right.$
$\left.(\pi - 1)\right] + \frac{1}{2!}\left[(x-1)^2(0) + (y-\pi/2)^2(2) + 2(x-1)\right.$
$\left.(y - \pi/2)(\pi - 1)\right]$

MACLAURIN SERIES (One variables)

A function $f(x)$ is defined at the point $a=0$.

$$f(x) = f(0) + \frac{(x-0)}{1!} f'(0) + \frac{(x-0)^2}{2!} f''(0) + \dots + \frac{(x-0)^n}{n!} f^n(0)$$

i.e.

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0).$$

Problems

1) Using maclaurin series of expand $e^{x.\sin x}$ up to the term x^3.

Soln: $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$ — (1)

given. $f(x) = e^{x.\sin x}$ — (2)

$x=0$.

$f(0) = e^{0 \sin 0}$

$f(0) = 1$

Diff eq (2) w.r.t x

$f'(x) = e^{x.\sin x} \cdot x \cos x + \sin x.(1)$ — (3)

$f'(0) = e^{0 \sin 0} \cdot 0 \cos 0 + \sin 0$

$f'(0) = 1 . 0 . 1 + 0$

$f'(0) = 0$

Diff eq (3) w.r.t x

$\cancel{f'(x) = e^{x\sin x} [x(-\sin x) + \cos x(1)]}$

$f'(x) = f(x) . (x \cos x + \sin x)$

$f''(x) = f(x)[x(-\sin x) + \cos x (1) + \cos x] + (x \cos x + \sin x) f'(x)$ — (4)

$x=0$

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$$f''(0) = f(0)[0(-\sin 0) + \cos 0\,(1) + \cos 0] + [0 \cos 0 + \sin 0]\, f'(0)$$

$$f''(0) = 1[1+1]$$

$$f''(0) = 2$$

Diff eq^n (4) w.r.t x.

$$f'''(x) = f(x)[x(-\cos x) + (-\sin x)(1) - \sin x - \sin x] + [-x \sin x + \cos x + \cos x] f'(x) + (x \cos x + \sin x) f''(x) + f'(x)[x(-\sin x) + \cos x\,(1) + \cos x].$$

$$f'''(0) = f(0)[0(-\cos 0) + (-\sin 0)(1) - \sin 0 - \sin 0] + [-0 \sin 0 + \cos 0 + \cos 0] f'(0) + (0 \cos 0 + \sin 0) f''(0) + f'(0)[0(-\sin 0) + \cos 0\,(1) + \cos 0]$$

$$f'''(0) = 1[(1+1)0]$$

$$f'''(0) = 0$$

Substituting all values in eq^n (1)

$$f(x) = 1 + \frac{x}{1}(0) + \frac{x^2}{2}(2) + \frac{x^3}{6}(0)$$

$$f(x) = 1 + x^2$$

2] Using maclaurin series of expand $\sin^{-1}x$ up to the term containing x^4.

Solu :- $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)$ — (1)

consider $f(x) = \sin^{-1}(x)$ — (2)

at $x=0$. $f(0) = \sin^{-1}(0)$

$$f(0) = 0$$

Diff eq^n (2) w.r.t x

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{--- (3)}$$

$$f'(0) = \frac{1}{\sqrt{1-0^2}} = \frac{1}{\sqrt{1}}$$

$$f'(0) = 1$$

Diff eq^n (3) w.r.t x

$$f'(x) \cdot \sqrt{1-x^2} = 1$$

square on b.s

$$[f'(x)]^2 (1-x^2) = 1$$

$$[f'(x)]^2 (-2x) + (1-x^2) \cdot 2f'(x)^{2-1} \cdot f''(x) = 0$$

$$(f'(x))^2 (-2x) + (1-x^2)\, 2f'(x) f''(x) = 0 \quad \text{--- (4)}$$

$$(f'(0))^2 (-\cancel{2(0)})^{0} + (1-0^2)\, 2f'(0) f''(0) = 0.$$

$$(1)^2 (1)\, 2(1)\, f''(0) = 0$$

$$2f''(0) = 0$$

$$f''(0) = 0$$

Diff eq^n (4) w.r.t. x

3) Expand $e^{\sin x}$ using maclaurin's series upto the terms containing x^4.

Soln: $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0)$ — (1)

Let $f(x) = e^{\sin x}$ — (2)

$f(0) = e^{\sin 0} = e^0$

$f(0) = 1$

Diff eq (2) w.r.t x

$f'(x) = e^{\sin x} \cdot \cos x$

$f'(x) = f(x) \cdot \cos x$ — (3)

$f'(0) = f(0) \cdot \cos 0.$

$f'(0) = 1 \cdot 1 \quad \Rightarrow f'(0) = 1.$

Diff eq^n (3) w.r.t x

$f''(x) = f(x) \cdot (-\sin x) + \cos x \, f'(x)$ — (4)

$f''(0) = f(0) \cdot (-\sin 0) + \cos 0 \, f'(0)$

$f''(0) = 1$

Diff eq^n (4) w.r.t x

$f'''(x) = -[f(x) \cos x + \sin x \, f'(x)] + [\cos x \, f''(x) + f'(x) \cdot -\sin x]$ — (5)

$f'''(0) = -[f(0) \cos 0 + \cancel{\sin 0} \, f'(0)] + [\cos 0 \, f''(0) + f'(0) \cdot \cancel{-\sin 0}]$

$f'''(0) = -1 + 1$

$f'''(0) = 0$

Diff eq^n (5) w.r.t x

$f^{iv}(x) = -[(f(x) \, {-\sin x} + \cos x \, f'(x)) + (\sin x \, f''(x) + f'(x) \cos x)] + [(\cos x \, f'''(x) + f''(x)(-\sin x)) + (f'(x) \, {-\cos x}) + (-\sin x \cdot f''(x))]$

$f^{iv}(0) = -[(f(0) \cancel{-\sin 0} + \cos 0 \, f'(0)) + (\cancel{\sin 0} \, f''(0) + f'(0) \cos 0)] + [(\cos 0 \, f'''(0) + f''(0) \cancel{(-\sin 0)}) + (f'(0) \, {-\cos 0}) + (\cancel{-\sin 0} \cdot f''(0))]$

$f^{iv}(0) = -1 - 1 + 0 - 1$ $\quad f^{iv}(0) = -3$

Substituting all the values in eq^n 1

$$f(x) = 1 + \frac{x}{1}(1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(0) + \frac{x^4}{\cancel{24}\,8}(-3)$$

$$f(x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

4) using maclaurin's series, prove that

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} \cdots$$

Sol^n: $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)$

Let $f(x) = \sqrt{1+\sin 2x}$ — 1

$= \sqrt{\cos^2 x + \sin^2 x + 2\cos x \cdot \sin x}$

$= \sqrt{(\cos x + \sin x)^2}$

$f(x) = \cos(x) + \sin(x)$ — 2

$f(0) = \cos 0 + \sin 0$

$f(0) = 1$

Diff eq^n 2 w.r.t x

$f'(x) = -\sin x + \cos x$ — 3

$f'(0) = -\sin 0 + \cos 0$

$f'(0) = 1$

Diff eq^n 3 w.r.t x

$f''(x) = -\cos x - \sin x$ — 4

$f''(0) = -\cos 0 - \sin 0$

$f''(0) = -1$

Diff eq^n 4 w.r.t x.

$f'''(x) = \sin x - \cos x$ — 5

$f'''(0) = \sin 0 - \cos 0$

$f'''(0) = \underline{-1}$

Diff eq^n 5 w.r.t x

$f^{iv}(x) = \cos x + \sin x$

$f^{iv}(0) = \cos 0 + \sin 0$

$f^{iv}(0) = \underline{1}$

Substituting all the values in eq^n 1

$$f(x) = 1 + \frac{x}{1}(-1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(-1) + \frac{x^4}{24}(1)$$

$$f(x) = 1 - x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

5] Expand $\log(\sec x)$ up to the term containing x^4 using maclaurin's series.

OR

Expand $\log(\sec x)$ in ascending powers of x up to the first three terms.

Soln:- $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)$ — 1

Let $f(x) = \log(\sec x)$ — 2

$f(0) = \log(\sec 0)$

$f(0) = \log 1 \Rightarrow \underline{f(0) = 0}$

Diff eq^n 2 w.r.t x

$$f'(x) = \frac{1}{\cancel{\sec x}}(\cancel{\sec x}\cdot\tan x)$$

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$f'(x) = \tan x$ — (3)

$f'(0) = \tan 0 \Rightarrow f'(0) = 0$

Diff eqⁿ (3) w.r.t x

$f''(x) = \sec^2 x$ — (4)

$f''(0) = \sec^2 0 \Rightarrow f''(0) = 1$

Diff eqⁿ (4) w.r.t x

$f''(x) = \sec^2 x$

$f''(x) = 1 + \tan^2 x$

$f''(x) = 1 + (f'(x))^2$

$f'''(x) = 2 f'(x) f''(x)$ — (5)

$f'''(0) = 2 f'(0) f''(0)$

$f'''(0) = 2(0)(1) \Rightarrow f'''(0) = 0$

Diff eqⁿ (5) w.r.t x

$f^{iv}(x) = 2[f'(x) f'''(x) + f''(x) f''(x)]$

$f^{iv}(0) = 2[f'(0) f'''(0) + f''(0) f''(0)]$

$f^{iv}(0) = 2[0 \cdot 0 + 1 \cdot 1]$

$f^{iv}(0) = 2[1] \Rightarrow f^{iv}(0) = 2$

Substituting all the values in eqⁿ (1)

$f(x) = 0 + \frac{x}{1}(0) + \frac{x^2}{2}(1) + \frac{x^3}{6}(0) + \frac{x^4}{\cancel{24}\,12}(\cancel{2})$

$f(x) = \frac{x^2}{2} + \frac{x^4}{12}$

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6] Expand $\log(1+x)$ up to the x^5 terms.

Sol:- $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0)$ — (1)

Let $f(x) = \log(1+x)$ — (2)

$f(0) = \log(1+0)$

$f(0) = \log 1 \Rightarrow f(0) = 0$

Diff eq^n (2) w.r.t x

$f'(x) = \frac{1}{1+x}(1)$ — (3)

$f'(0) = \frac{1}{1+0} \Rightarrow f'(0) = 1$

Diff eq^n (3) w.r.t x.

$f''(x) = \frac{-1}{(1+x)^2}$ — (4)

$f''(0) = \frac{-1}{(1+0)^2} \Rightarrow f''(0) = -1$

Diff eq^n (4) w.r.t x.

$f'''(x) = \frac{2}{(1+x)^3}$ — (5)

$f'''(0) = \frac{2}{(1+0)^3} \Rightarrow f'''(0) = 2$

eq^n (4)

$f''(x) = -(1+x)^{-2}$

$= -(-2(1+x)^{-2-1}(1)$

$= 2(1+x)^{-3}$

$= \frac{2}{(1+x)^3}$

Diff eq^n (5) w.r.t. x

$f^{iv}(x) = \frac{-6}{(1+x)^4}$ — (6)

$f^{iv}(0) = \frac{-6}{(1+0)^4}$

$f^{iv}(0) = -6$

Diff eq^n (6) w.r.t x

$f^{v}(x) = \frac{24}{(1+x)^5}$

$f^{v}(0) = \frac{24}{(1+0)^5}$

$f^{v}(0) = 24$

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Substituting all the values in eq^n (1).

$$f(x) = 0 + \frac{x}{1}(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \frac{x^5}{120}(24)$$

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{5}$$

7] Expand $\log(1+e^x)$ up to the fourth degree.

Soln :- $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)$ — (1)

Let $f(x) = \log(1+e^x)$ — (2)

$f(0) = \log(1+e^0) \Rightarrow f(0) = \log 2$

Diff eq^n (2) w.r.t. x.

$$f'(x) = \frac{1}{1+e^x} \cdot e^x \quad \text{— (3)}$$

$$f'(0) = \frac{e^0}{1+e^0} \Rightarrow f'(0) = \frac{1}{1+1}$$

$$f'(0) = 1/2$$

from eq^n (3). $f'(x)(1+e^x) = e^x$

Diff w.r.t. x.

$$f'(x)(e^x) + (1+e^x)f''(x) = e^x \quad \text{— (4)}$$

$$f'(0)(e^0) + (1+e^0)f''(0) = e^0$$

$$(1/2)(1) + (1+1)f''(0) = 1$$

$$1/2 + 2f''(0) = 1$$

$$2f''(0) = 1 - 1/2 \Rightarrow f''(0) = 1/2 \times 1/2$$

$$f''(0) = 1/4$$

Diff eq^n (4) w.r.t x

$$f'(x)e^x + e^x f''(x) + (1+e^x) f'''(x) + f''(x) e^x = e^x \quad \text{--- (5)}$$

$$f'(0)e^0 + e^0 f''(0) + (1+e^0) f'''(0) + f''(0)e^0 = e^0$$

$$(1/2)(1) + 1(1/4) + (1+1) f'''(0) + (1/4)(1) = 1$$

$$2f'''(0) = 1 - 1/2 - 1/4 - 1/4 \qquad \frac{4-2-1-1}{4} = 0$$

$$2f'''(0) = 0 \implies \underline{f'''(0) = 0}$$

Diff eq^n (5) w.r.t x

$$f'(x)e^x + e^x f''(x) + e^x f'''(x) + f''(x)e^x + (1+e^x) f^{iv}(x) + f'''(x)$$
$$(1+e^x) + f''(x)e^x + e^x f'''(x) = e^x.$$

$$f'(0)e^0 + e^0 f''(0) + e^0 f'''(0) + f''(0)e^0 + (1+e^0) f^{iv}(0) + f'''(0)$$
$$(1+e^0) + f''(0)e^0 + e^0 f'''(0) = e^0$$

$$(1/2)(1) + (1)(1/4) + \cancel{(1)(0)} + (1/4)(1) + (1+1) f^{iv}(0) + (0)$$
$$\cancel{(1+1)} + (1/4)(1) + \cancel{(1)(0)} = 1$$

$$2f^{iv}(0) = 1 - 1/2 - 1/4 - 1/4 - 1/4 \qquad \frac{4-2-1-1-1}{4}$$

$$4 - 5 = -1/4$$

$$2f^{iv}(0) = -1/4$$

$$\underline{f^{iv}(0) = -1/8}$$

Substituting all the values in eq^n (1).

$$f(x) = \log 2 + \frac{x}{1}(1/2) + \frac{x^2}{2}(1/4) + \cancel{\frac{x^3}{6}(0)} + \frac{x^4}{24}(-1/8)$$

$$\underline{f(x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192}}$$

Maclaurin series for function of two variables

1] Expand e^{ax+by} about the point (0.0) upto the 3^{rd} degree.

OR

Expand e^{ax+by} in the powers of x & y up to ~~the~~ 3^{rd} term.

Soln: $f(x.y) = f(0.0) + \frac{1}{1!}[x.f_x(0.0) + y f_y(0.0)] + \frac{1}{2!}[x^2 f_{xx}(0.0) + y^2 f_{yy}(0.0) + 2x.y f_{xy}(0.0)] + \frac{1}{3!}[x^3 f_{xxx}(0.0) + y^3 f_{yyy}(0.0) + 3x^2y f_{xxy}(0.0) + 3xy^2 f_{xyy}(0.0)]$ → (1)

Let $f(xy) = e^{ax+by}$

$f(0.0) = e^{a.0+b.0}$

$= e^0$

$f(0.0) = 1$

$f_x(x.y) = e^{ax+by} . a$

$f_x(0.0) = e^{a.0+b.0} . a$

$f_x(0.0) = a$

$f_{xx}(x.y) = e^{ax+by} . a^2$

$f_{xx}(0.0) = e^{a.0+b0} . a^2$

$f_{xx}(0.0) = a^2$

$f_{xxx}(x.y) = e^{ax+by} . a^3$

$f_{xxx}(0.0) = a^3$

$f_y(x.y) = e^{ax+by} . b$

$f_y(0.0) = e^{a.0+b.0} . b$

$f_y(0.0) = b$

$f_{yy}(0.0) = e^{ax+by} b^2$

$f_{yy}(0.0) = e^{a0+b.0} . b^2$

$f_{yy}(0.0) = b^2$

$f_{yyy}(x.y) = e^{ax+by} b^3$

$f_{yyy}(0.0) = e^{a.0+b.0} . b^3$

$f_{yyy}(0.0) = b^3$

$$f_{xxy}(x.y) = \frac{\delta}{\delta x}\left[\frac{\delta^2 u}{\delta x . \delta y}\right]$$

$$= \frac{\delta}{\delta x}\left[e^{ax+by} . b\right]$$

$$f_{xy}(x.y) = e^{ax+by} . a.b$$

$$f_{xy}(0.0) = e^0 . ab \implies f_{xy}(0.0) = ab$$

$$f_{xxy}(x.y) = \frac{\delta}{\delta x}\left[\frac{\delta^2 u}{\delta x . \delta y}\right]$$

$$= \frac{\delta}{\delta x}\left[e^{ax+by} . ab\right]$$

$$= \frac{\delta}{\delta x}\left[e^{ax+by} . a^2 b\right]$$

$$f_{xxy}(0.0) = a^2 b$$

$$f_{xyy}(x.y) = \frac{\delta}{\delta x}\left[\frac{\delta^2 u}{\delta y^2}\right]$$

$$= \frac{\delta}{\delta x}\left[e^{ax+by} . b^2\right]$$

$$f_{xyy}(x.y) = e^{ax+by} . ab^2$$

$$f_{xyy}(0.0) = ab^2$$

Substitute all the values in eq ①

$$f(x.y) = 1 + 1[x.a + yb] + \frac{1}{2}[x^2 a^2 + y^2 b^2 + 2xy\,ab] + \frac{1}{6}[x^3 a^3 + y^3 b^3 + 3x^2 y a^2 b + 3xy^2 ab^2]$$

2] Expand $e^x \log(1+y)$ about the origin of in the powers of x & y upto the 3rd degree.

Soln :- $f(x,y) = f(0,0) + \frac{1}{1!}[x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0)]$. — (1)

$f(x,y) = e^x \log(1+y)$ $\therefore f(0,0) = e^0 \cdot \log(1+0)$

$f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y)$ $\cdot f_x(0,0) = e^0 \log(1+0)$

$f_x(0,0) = 0$

$f_{xx}(x,y) = e^x \log(1+y)$ $f_{xx}(0,0) = e^0 \log(1+0)$

$f_{xx}(0,0) = 0$

$f_{xxx}(x,y) = e^x \log(1+y)$. $f_{xxx}(0,0) = e^0 \log(1+0)$

$f_{xxx}(0,0) = 0$

$f_y(x,y) = e^x \frac{1}{1+y}(1)$ $f_y(0,0) = e^0 \cdot \frac{1}{1+0} = 1$

$f_{yy}(x,y) = \frac{-e^x}{(1+y)^2}$ $f_{yy}(0,0) = \frac{-e^0}{(1+0)^2} = -1$

$f_{yyy}(x,y) = \frac{2e^x}{(1+y)^3}$ $f_{yyy}(0,0) = \frac{2e^0}{(1+0)^2} = 2$

$f_{xy}(x,y) = \frac{\partial}{\partial x}\left[e^x \frac{1}{1+y}(1)\right]$

$f_{xy}(x,y) = e^x \frac{1}{1+y}$ $f_{xy}(0,0) = e^0 \frac{1}{1+0}$

$f_{xy}(0,0) = 1$

$f_{xxy}(x,y) = \frac{\delta}{\delta x}\left[\frac{e^x}{1+y}\right]$

$f_{xxy}(x.y) = \frac{e^x}{1+y}$ $\qquad$ $f_{xxy}(0.0) = \frac{e^0}{1+0}$

$f_{xxy}(0.0) = 1$

$f_{xyy}(x.y) = \frac{\delta}{\delta x}\left[\frac{-e^x}{(1+y)^2}\right]$

$f_{xyy}(x.y) = \frac{-e^x}{(1+y)^2}$ $\qquad$ $f_{xyy}(0.0) = \frac{-e^0}{(1+0)^2}$

$f_{xyy}(0.0) = -1$

Substitute all the values in eq ①.

$f(x.y) = 0 + \frac{1}{1!}[x.\cancel{0} + y.1] + \frac{1}{2}[\cancel{x^2.0} + y^2(-1) + 2xy(1)] + \frac{1}{3!}[\cancel{x^3(0)} + y^3(2) + 3x^2y(1) + 3xy^2(-1)]$

$f(x.y) = y + \frac{1}{2}[-y^2 + 2xy] + \frac{1}{6}[2y^3 + 3x^2y - 3xy^2].$

Extreme values of a function of two variables

The necessary condition for $f(x, y)$ to have maximum or minimum value at (a, b) is that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Here the point (a, b) is known as stationary point or critical point.

Let $f_{xx}(a, b) = A$, $f_{xy}(a, b) = B$, $f_{yy}(a, b) = C$

① $f(a, b)$ is maximum value if $AC - B^2 > 0$ and $A < 0$.

② $f(a, b)$ is a minimum value if $AC - B^2 > 0$ and $A > 0$.

③ $f(a, b)$ is not an extreme if $AC - B^2 < 0$ [Extreme = neither minimum nor maximum] and ~~this~~ case the point (a, b) is called the saddle point.

④ If $AC - B^2 = 0$, it means further consideration.

Problems:

1) Find the extreme values of the function.

$f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

Sol: consider $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

$f_x(x, y) = 3x^2 - 3$ $\qquad$ $f_y(x, y) = 3y^2 - 12$

$f_{xx}(x, y) = 6x$ $\qquad$ $f_{yy}(x, y) = 6y$

$$f_{xy}(x, y) = \frac{\delta^2}{\delta x}\left[\frac{\delta u}{\delta y}\right]$$

$$f_{xy}(x.y) = \frac{\partial}{\partial x}[3y^2-12]$$

$$f_{xy}(x.y) = 0$$

We shall find points (x.y) such that $\cancel{f_x=0}$ & $f_y=0$

$f_x = 0$

$3x^2 - 3 = 0$

$3(x^2-1) = 0$

$x^2 - 1 = 0$

$x = \pm 1$

$f_y = 0$

$3y^2 - 12 = 0$

$3(y^2-4) = 0$

$y^2 - 4 = 0$

$y = \pm 2$

$\therefore$ (1,2), (1,-2), (-1,2), (-1,-2) are the stationary or critical points.

points	(1,2)	(1,-2)	(-1,2)	(-1,-2)
$A = f_{xx} = 6x$	$6(1) = 6 > 0$ min	6	-6	$-6 < 0$ max
$B = f_{xy} = 0$	0	0	0	0
$C = f_{yy} = 6y$	$6(2) = 12$	-12	12	-12
$AC - B^2$	$12(6) - 0^2 = 72 > 0$	$-72 < 0$ saddle	$-72 < 0$	$72 > 0$
conclusion	minimum points	saddle point	saddle point	maximum point.

maximum value of $f(x.y)$ is

$$f(-1,-2) = -1 - 8 + 3 + 24 + 20$$

$$f(-1,-2) = 38$$

minimum value of $f(x.y)$ is

$$f(1,2) = 1 + 8 - 3 - 24 + 20$$

$$f(1,2) = 2$$

2]. Find the extreme values of the function $f(x.y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Soln :- Consider $f(x.y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

$f_x(x.y) = 3x^2 + 3y^2 - 6x$ $\quad f_y(x.y) = 6xy - 6y$

$f_{xx}(x.y) = \underline{6x - 6}$ $\quad f_{yy}(x.y) = \underline{6x - 6}$

$f_{xy} = \frac{\partial}{\partial x}[6xy - 6y]$

$f_{xy} = \underline{6y}$

We shall find points $(x.y)$ such that

$f_x = 0$ $\quad f_y = 0$

$3x^2 + 3y^2 - 6x = 0$ $\quad 6xy - 6y = 0$

$3(x^2 + y^2 - 2x) = 0$ $\quad 6y(x-1) = 0$ $\quad 6y = 0$

$x^2 + y^2 - 2x = 0$ $\quad x - 1 = 0$, $y = 0$

$\underline{y=0}$, $x^2 - 2x = 0$ $\quad \underline{x = 1}$, $\underline{y = 0}$

$x(x-2) = 0$ $\quad \underline{x=1}$, $1 + y^2 - 2 = 0$

$\underline{x = 0 \ , \ x = 2}$ $\quad y^2 - 1 = 0$

$\underline{y = \pm 1}$

$\therefore (0,0), (2,0), (1,1), (1,-1)$ are critical points

points	(0.0)	(2.0)	(1,1)	(1,-1)
$A = f_{xx} = 6x-6$	$-6<0$	$6(2)-6 = 6>0$	0	0
$B = f_{xy} = 6y$	0	0	6	-6
$C = f_{yy} = 6x-6$	-6	6	0	0
$AC - B^2$	$36>0$	$36>0$	$0-36 = -36<0$	$-36<0$
conclusion	maximum point	minimum point	saddle point	saddle point

maximum value of $f(x.y)$ is

$f(0.0) = 4$

minimum value of $f(x.y)$ is

$f(2.0) = 0$

3] Find the maximum and minimum values of the function. $x^3+3xy^2-15x^2-15y^2+72x$.

Solu:- Let $f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$

$f_x(x,y) = 3x^2+3y^2-30x+72$, $f_y(x,y) = 6xy-30y$

$f_{xx}(x,y) = 6x-30$ $f_{yy}(x,y) = 6x-30$

$f_{xy} = f_x[6xy-30y]$

$f_{xy} = 6y$

we shall find points (x,y) such that

$f_x = 0$ $f_y = 0$

$3x^2+3y^2-30x+72 = 0$ $6xy-30y = 0$

$3[x^2+y^2-10x+24] = 0$ $6y(x-5) = 0$

$x^2+y^2-10x+24 = 0$ $6y = 0$ $x-5 = 0$

$y=0$, $x^2-10x+24 = 0$ $y = 0$, $x = 5$

$x = 4, 6$.

$\therefore (4.0), (6.0)$ are critical points

$x=5$. $25+y^2-50+24 = 0$

$y^2-1 = 0 \Rightarrow y = \pm 1$

$\therefore (5.1)\ (5,-1)$ are critical points

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Points	(4,0)	(6,0)	(5,1)	(5,-1)
$A = f_{xx} = 6x-6$	$-6<0$	$6>0$	0	0
$B = f_{xy} = 6y$	0	0	6	-6
$C = f_{yy} = 6x-6$	-6	6	0	0
$AC - B^2$	$36>0$	$36>0$	$-36<0$	$-36<0$
conclusion.	max. point	min point	Saddle point	Saddle point

maximum value of $f(x,y)$ is.

$$f(4,0) = 112$$

minimum value of $f(x,y)$ is

$$f(6,0) = 108$$

4]. Find the extreme values of $f(x,y) = x^3y^2(1-x-y)$

Soln:- Let. $f(x,y) = x^3y^2(1-x-y)$

$$f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$$

$f_x(x,y) = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$. $f_y(x,y) = 2x^3y - 2x^4y - 3x^3y^2$

$f_{xx}(x,y) = 6xy^2 - 12x^2y^2 - 6xy^3$. $\cancel{f_{yy}(x,y) = 6x^2y - 8x^3}$

$$f_{yy}(x,y) = 2x^3 - 2x^4 - 6x^3y$$

$$f_{xy} = f_x[2x^3y - 2x^4y - 3x^3y^2]$$

$$f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$$

We shall find points (x,y) such that

$f_x = 0$	$f_y = 0$
$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$	$2yx^3 - 2x^4y - 3x^3y^2 = 0$
$x^2y^2[3-4x-3y] = 0$	$x^3y[2-2x-3y] = 0$
$x=0,\ y=0.\ 3-4x-3y=0$	$x=0,\ y=0;\ 2-2x-3y=0$
$4x+3y=3$	$2x+3y=2$

$\left.\begin{matrix}x=0\\y=0\end{matrix}\right\}$ $\left.\begin{matrix}x=0\\2x+3y=2\end{matrix}\right\}$ $\left.\begin{matrix}y=0\\2x+3y=2\end{matrix}\right\}$ $\left.\begin{matrix}x=0\\y=0\end{matrix}\right\}$ $\left.\begin{matrix}x=0\\4x+3y=3\end{matrix}\right\}$ $\left.\begin{matrix}y=0\\4x+3y=3\end{matrix}\right\}$

$y=2/3$. $x=1$ $y=1$ $x=3/4$

$\therefore (0,0), (0, 2/3), (1,0), (0,1), (3/4, 0)$ are the critical points.

$$4x+3y=3$$
$$2x+3y=2$$
$$(+)\ (-)\ \ (-)$$
$$2x=1$$
$$x=1/2$$

Substitute $x=1/2$

$$4(1/2)+3y=3$$
$$3y=3-2$$
$$y=1/3$$

$\therefore (1/2, 1/3)$ are also critical points

Also $A=f_{xx}=6xy^2-12x^2y^2-6xy^3=6xy^2(1-2x-y)$

$B=f_{xy}=6x^2y-8x^3y-9x^2y^2=x^2y(6-8x-9y)$

$C=f_{yy}=2x^3-2x^4-6x^3y=2x^3(1-x-3y)$

It is evident that either $A=0$ or $C=0$ or both A and C are zero in respect of all the stationary points except $(1/2, 1/3)$

$$A=6(1/2)(1/3)^2[1-2(1/2)-1/3]$$
$$=3\times 1/9_3[1-1-1/3]$$
$$A=-1/9<0.$$

$$B=(1/2)^2(1/3)[6-8(1/2)-9(1/3)]$$
$$=1/4\times 1/3[6-1]$$
$$B=-1/12$$

$$C=2(1/2)^3[1-1/2-3(1/3)]$$

$C = 2 \times \frac{1}{8} \left[-\frac{1}{2}\right]$

$C = -\frac{1}{8}$

$\therefore AC - B^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2$

$= \frac{1}{72} - \frac{1}{144}$

$AC - B^2 = \frac{1}{144} > 0$

But $A = -\frac{1}{9} < 0$

Hence $\left(\frac{1}{2}, \frac{1}{3}\right)$ is a maximum point.

Thus maximum value of $f(x, y)$ is

$f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left[1 - \frac{1}{2} - \frac{1}{3}\right]$

$= \frac{1}{8} \times \frac{1}{9} \left[\frac{1}{6}\right]$

$f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432}$

5]. Find the extreme values of $f(x, y) = \sin x + \sin y + \sin(x+y)$.

Soln: Let $f(x, y) = \sin x + \sin y + \sin(x+y)$

$f_x(x, y) = \cos x + \cos(x+y)$ (1)

$f_y(x, y) = \cos y + \cos(x+y)$

$f_{xx}(x, y) = -\sin x - \sin(x+y)$

$f_{yy}(x, y) = -\sin y - \sin(x+y)$

$f_{xy}(x, y) = f_x(\cos y + \cos(x+y))$

$f_{xy} = -\sin(x+y)$

we shall find points (x, y) such that

$f_x = 0$ $\qquad$ $f_y = 0$

$\cos x + \cos(x+y) = 0$ — (1) $\qquad$ $\cos y + \cos(x+y) = 0$ — (2)

$\cos(x+y) = -\cos x$ $\qquad$ $\cos(x+y) = -\cos y$

Thus $-\cos x = -\cos y$

$$\underline{x = y}$$

from eq (1) $\cos x + \cos(x+x) = 0$

$\cos x + \cos 2x = 0$ $\qquad$ $[\cos 2x = 2\cos^2 x - 1]$

i.e. $\cos x + (2\cos^2 x - 1) = 0$

$\cancel{\cos x + (2\cos^2 x\ldots}$ $2\cos^2 x + \cos x + \cos x - \cos x - 1 = 0$

$2\cos^2 x + 2\cos x - \cos x - 1 = 0$ factorisation.

$2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$

$(\cos x + 1)(2\cos x - 1) = 0$

$\cos x = -1$ $\qquad$ $2\cos x = 1$

$x = \cos^{-1}(-1)$ $\qquad$ $x = \cos^{-1}(1/2)$ $\qquad$ $y = x$

$\underline{\underline{x = \pi}}$ $\qquad$ $\underline{\underline{x = \pi/3}}$ $\qquad$ $y = \pi$, $\underline{\underline{y = \pi/3}}$

Thus (π, π) $(\pi/3, \pi/3)$ are the critical points

Points	(π, π)	$(\pi/3, \pi/3)$
$A = f_{xx} = -\sin x - \sin(x+y)$	0	$-\sqrt{3} < 0$
$B = f_{xy} = -\sin(x+y)$	0	$-\sqrt{3}/2$
$C = f_{yy} = -\sin y - \sin(x+y)$	0	$-\sqrt{3}$
$AC - B^2$	0	$9/4 > 0$
Conclusion.	further consideration	maximum points

$-\sin \pi/3 - \sin 2\pi/3$

$-\sqrt{3}/2 - \sqrt{3}/2 = \frac{-\sqrt{3}-\sqrt{3}}{2} = \frac{-2\sqrt{3}}{\cancel{2}}$

$AC - B^2 = (-\sqrt{3})(-\sqrt{3}) - (-\sqrt{3}/2)^2$

$= (\sqrt{3})^2 - 3/4 = \frac{12-3}{4}$

$= 9/4$

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maximum value of $f(x, y)$ is

$$f(\pi/3, \pi/3) = \sin(\pi/3) + \sin(\pi/3) + \sin(\pi/3 + \pi/3)$$

$$= \sqrt{3}/2 + \sqrt{3}/2 + \sqrt{3}/2$$

$$= \frac{\sqrt{3}+\sqrt{3}+\sqrt{3}}{2}$$

$$f(\pi/3, \pi/3) = \frac{3\sqrt{3}}{2}$$

6] Show that $z(x, y) = x^3 + y^3 - 3xy + 1$ is minimum at $(1, 1)$

Solu :- $z(x, y) = x^3 + y^3 - 3xy + 1$.

$z_x(x, y) = 3x^2 - 3y$ $z_y(x, y) = 3y^2 - 3x$

$z_{xx}(x, y) = 6x$ $z_{yy}(x, y) = 6y$

$$z_{xy}(x, y) = z_x(3y^2 - 3x)$$

$$z_{xy}(x, y) = -3$$

At $(1, 1)$

$A = z_{xx} = 6(1) = 6$ $C = f_{yy} = 6(1) = 6$

$B = z_{xy} = -3$

$$AC - B^2 = (6)(6) - (-3)^2$$

$$AC - B^2 = 27.$$

$\therefore A = 6 > 0$ and $AC - B^2 = 27 > 0$

$\therefore z(x, y)$ at $(1, 1)$ satisfy the necessary and sufficient conditions for minima.

Thus $z(x, y)$ is minimum at $(1, 1)$