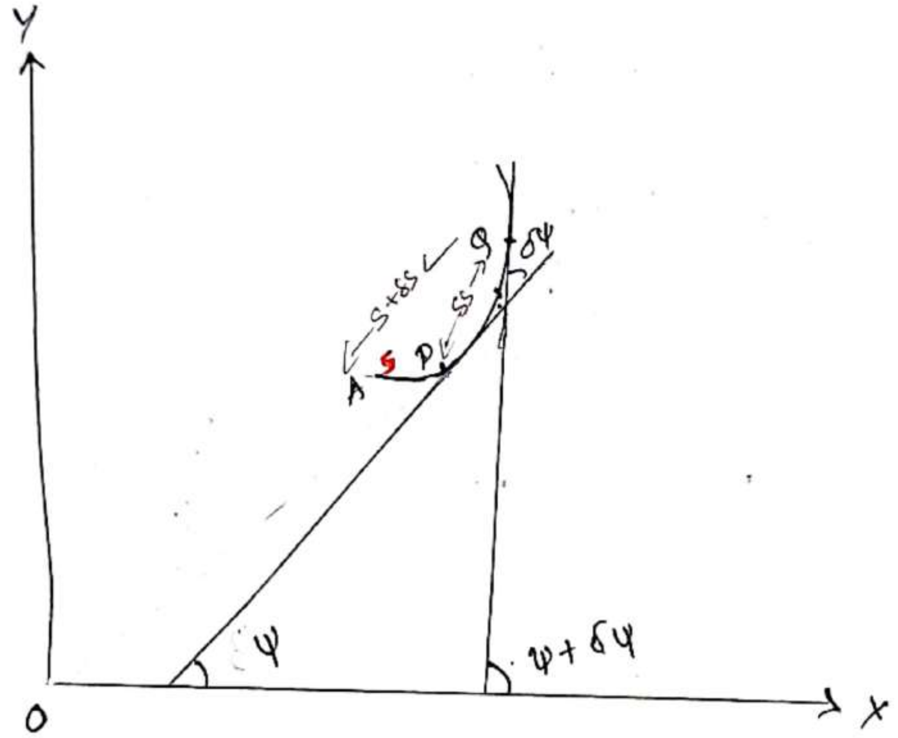
Curvature and Radius of curvature:

Curvature: Measuring the bendness of the curve.

(or)

The amount of bending of a curve at given point is called curvature.



$\widehat{AP} = s$ and $\widehat{AQ} = s+\delta s$ so that $\widehat{PQ} = \delta s$

Let ψ and $\psi+\delta\psi$ respectively be the $\angle$'es made by the tangent at P and Q with the X-axis

The angle $\delta\psi$ b/w the tangents is called the bending of the curve.

$$\text{Curvature} = K = \frac{d\psi}{ds}$$

If $K \neq 0$ the reciprocal of the curvature is called as the radius of curvature and denoted by ρ

Radius of curvature $= \rho = \frac{1}{K} = \frac{ds}{d\psi}$

Expression for the radius of curvature for a Cartesian curve.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}.$$

where, $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

Note:

We always take the sign of K and ρ to be positive.

Find the radius of curvature for the curve whose intrinsic equation is $s = a \log \tan\left(\pi/4 + \psi/2\right)$

Sol^n:

$s = a \log \tan\left(\pi/4 + \psi/2\right)$ and we have $\rho = \dfrac{ds}{d\psi}$

Differentiating w.r.t ψ

$$\frac{ds}{d\psi} = a \cdot \frac{1}{\tan\left(\pi/4 + \psi/2\right)} \cdot \sec^2\left(\pi/4 + \psi/2\right) \cdot \frac{1}{2}$$

$$= \frac{a}{2} \frac{\cos\left(\pi/4 + \psi/2\right)}{\sin\left(\pi/4 + \psi/2\right)} \cdot \frac{1}{\cos^2\left(\pi/4 + \psi/2\right)}$$

$$= \frac{a}{2 \sin\left(\pi/4 + \psi/2\right) \cos\left(\pi/4 + \psi/2\right)}$$

But $\boxed{2 \sin\theta \cos\theta = \sin 2\theta}$

$$\therefore \frac{ds}{d\psi} = \frac{a}{\sin\left[2\left(\pi/4 + \psi/2\right)\right]}$$

$$= \frac{a}{\sin\left(\pi/2 + \psi\right)}$$

$$= \frac{a}{\cos\psi}$$

$$\boxed{\rho = a \sec\psi}$$

→ S.T the radius of curvature for the catenary of uniform strength $y = a \log \sec(x/a)$ is $a \sec(x/a)$

Sol^n: We have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$.

Now consider,

$$y = a \log \sec(x/a)$$

$$\frac{dy}{dx} = y_1 = \frac{\cancel{a}}{\cancel{\sec(x/a)}} \cdot \cancel{\sec(x/a)} \tan(x/a) \cdot \frac{1}{\cancel{a}}.$$

$$\boxed{y_1 = \tan(x/a)}$$

Also $\boxed{y_2 = \frac{1}{a} \sec^2(x/a)}$

$$\therefore \rho = \frac{\left[1+\tan^2(x/a)\right]^{3/2} \cdot a}{\sec^2(x/a)}$$

$$= \frac{a\left[\sec^2(x/a)\right]^{3/2}}{\sec^2(x/a)}$$

$$= \frac{a \sec^3(x/a)}{\cancel{\sec^2}(x/a)}$$

$$\boxed{\rho = a \sec(x/a)}$$

$\Rightarrow$ S.T for the Catenary $y = c\cosh(x/c)$ the radius of curvature is equal to y^2/c. [which is also equal to the length of the normal intercepted b/n the curve & the x-axis] ×

Sol: we have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$y = c\cosh\left(\frac{x}{c}\right)$$

$$y_1 = c.\sinh\left(\frac{x}{c}\right).\frac{1}{c} = \sinh\left(\frac{x}{c}\right)$$

$$y_2 = \frac{1}{c}\cosh\left(\frac{x}{c}\right)$$

$$\rho = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{3/2}.c}{\cosh\left(\frac{x}{c}\right)} = \frac{c\left[\cosh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)}$$

$$\rho = \frac{c\cosh^3(x/c)}{\cosh(x/c)} = c\cosh^2(x/c)$$

$$\left\langle \rho = c\cosh^2(x/c) \right\rangle$$

But $\dfrac{y}{c} = \cosh\left(\dfrac{x}{c}\right)$

$$\rho = \cancel{c}.\frac{y^2}{c^{\cancel{2}}}$$

$$\left\langle \rho = y^2/c \right\rangle$$

→ Find the Radius of Curvature for the Curve $y = ax^2 + bx + c$ at $x = \frac{1}{2a}\left[\sqrt{a^2-1} - b\right]$.

$$y = ax^2 + bx + c$$

$$y_1 = 2ax + b, \quad y_2 = 2a$$

At the given Pt, $y_1 = \cancel{2a} \cdot \frac{1}{\cancel{2a}}\left[\sqrt{a^2-1} - b\right] + b$

$$y_1 = \sqrt{a^2-1}$$

$$y_2 = 2a$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\rho = \frac{\left[1 + (a^2-1)\right]^{3/2}}{2a}$$

$$\rho = \frac{(a^2)^{3/2}}{2a}$$

$$\left\langle \rho = \frac{a^2}{2} \right\rangle$$

Scanned with OKEN Scanner

Find the radius of curvature for the Folium of De-Cartes $x^3+y^3=3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on it.

Solⁿ:

$$x^3+y^3=3axy$$

Diff w.r.t x

$$3x^2+3y^2\frac{dy}{dx}=3a\left[x\frac{dy}{dx}+y\right]$$

i.e $$3(y^2-ax)\frac{dy}{dx}=3(ay-x^2)$$

$$\frac{dy}{dx}=y_1=\frac{ay-x^2}{y^2-ax}.$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $y_1=\dfrac{\frac{3a^2}{2}-\frac{9a^2}{4}}{\frac{9a^2}{4}-\frac{3a^2}{2}}$

$$\boxed{y_1=-1}$$

Now $$\frac{d^2y}{dx^2}=y_2=\frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

we have, $y^2-ax=\frac{9a^2}{4}-\frac{3a^2}{2}$ & $ay-x^2=\frac{3a^2}{2}-\frac{9a^2}{4}$

$=\frac{3a^2}{4}$ $\qquad$ $=-\frac{3a^2}{4}$

Hence $y_2 = \dfrac{\left(\frac{3a^2}{4}\right)(-a-3a) - \left(-\frac{3a^2}{4}\right)(-3a-a)}{\left(\frac{3a^2}{4}\right)^2}$

$$= \frac{-3a^3 - 3a^3}{\frac{9a^4}{16}} = \frac{16(-6a^3)}{9a^3}.$$

$$\boxed{y_2 = \frac{-32}{3a}}$$

we have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$\rho = \frac{(1+1)^{3/2}}{\frac{-32}{3a}} = \frac{(2)^{3/2}}{\frac{-32}{3a}} = \frac{\sqrt{8} \times 3a}{-32}$$

$$= \frac{2\sqrt{2} \cdot 3a}{-32}$$

$$= \frac{-3a}{8\sqrt{2}}$$

$$\boxed{\rho = \frac{3a}{8\sqrt{2}}}$$

→ Find the radius of curvature of the curve $x = a\log(\sec t + \tan t)$, $y = a\sec t$.

$x = a\log[\sec t + \tan t]$

$$\frac{dx}{dt} = \frac{a}{\sec t + \tan t} \cdot [\sec t.\tan t + \sec^2 t]$$

$$= \frac{a\sec t\,[\cancel{\sec t + \tan t}]}{\cancel{\sec t + \tan t}}$$

$$\frac{dx}{dt} = a\sec t$$

Also $y = a\sec t$

$$\frac{dy}{dt} = a\sec t.\tan t$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sec t.\tan t}{a\sec t} = \tan t$$

$y_1 = \tan t$ Diff. w.r.to x

$$y_2 = \sec^2 t \cdot \frac{dt}{dx}$$

$$y_2 = \sec^2 t \cdot \frac{1}{a\sec t} = \frac{\sec t}{a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2 t)^{3/2}}{\sec t} = \frac{a\sec^3 t}{\sec t}$$

$\rho = a\sec^2 t$

$\langle \rho = a\sec^2 t \rangle$

→ S.T the Radius of Curvature at any point θ on the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos\theta/2$.

Sol: $x = a(\theta + \sin\theta)$ $\qquad$ $y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = a(1+\cos\theta)$ $\qquad$ $\frac{dy}{d\theta} = a\sin\theta$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$$

$$y_1 = \frac{2\sin\theta/2 . \cos\theta/2}{2\cos^2\theta/2}$$

$$y_1 = \tan\theta/2$$

Diff w.r.to x we get

$$y_2 = \sec^2\theta/2 . \frac{1}{2} \frac{d\theta}{dx}$$

$$= \sec^2\frac{\theta}{2} . \frac{1}{2} . \frac{1}{a(1+\cos\theta)} = \frac{\sec^2\theta/2}{2a . 2\cos^2\theta/2}$$

$$y_2 = \frac{\sec^2\theta/2}{4a\cos^2\theta/2} \Rightarrow \frac{1}{4a}\sec^4(\theta/2)$$

$$\rho = \frac{[1+y_1^2]^{3/2}}{y_2} = \frac{[1+\tan^2(\theta/2)]^{3/2} . 4a}{\sec^4(\theta/2)}$$

$$= \frac{[\sec^2\theta/2]^{3/2} 4a}{\sec^4(\theta/2)} = \frac{4a . \sec^3(\theta/2)}{\sec^4\theta/2}$$

$$\rho = 4a\cos(\theta/2)$$

→ Find the Radius of Curvature of the tractrix $x = a[\cos t + \log \tan(t/2)]$, $y = a\sin t$.

$$x = a[\cos t + \log \tan(t/2)]$$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan(t/2)} \sec^2 t/2 \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\frac{\sin t/2}{\cos t/2}} \cdot \frac{1}{2\cos^2 t/2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\cos t/2 \cdot \sin t/2}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin 2\frac{t}{2}}\right] = a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{-\sin^2 t + 1}{\sin t}\right] = a \cdot \frac{\cos^2 t}{\sin t}$$

$$\frac{dx}{dt} = a\cos^2 t \cdot \operatorname{cosec} t, \qquad \frac{dy}{dt} = a\cos t$$

$$y_1 = \frac{dy/dt}{dx/dt} = \frac{a\cancel{\cos t}}{a\cos^{\cancel{2}} t \cdot \operatorname{cosec} t} = \frac{1}{\cos t \cdot \frac{1}{\sin t}}$$

$$\langle y_1 = \tan t \rangle$$

$$y_2 = \sec^2 t \cdot \frac{dt}{dx}$$

$$y_2 = \frac{\sec^2 t}{a\cos^2 t \cdot \operatorname{cosec} t} = \frac{\sec^4 t \cdot \sin t}{a}$$

$$\rho = \frac{(1 + \tan^2 t)^{3/2} \cdot a}{\sec^4 t \cdot \sin t} = \frac{a\sec^3 t}{\sec^4 t \cdot \sin t}$$

$$\langle \rho = a\cot t \rangle$$

→ Find the Radius of curvature of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \pi/4$

$x = a\cos^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta.\sin\theta$

$y = a\sin^3\theta$

$\frac{dy}{d\theta} = 3a\sin^2\theta.\cos\theta$

$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta.\cos\theta}{-3a\cos^2\theta.\sin\theta} = -\tan\theta$

$y_2 = -\sec^2\theta.\frac{d\theta}{dx} = \frac{-\sec^2\theta.}{-3a\cos^2\theta.\sin\theta} = \frac{\sec^4\theta.\mathrm{cosec}\,\theta}{3a}$

$\rho = \frac{(1+\tan^2\theta)^{3/2}.3a}{\sec^4\theta.\mathrm{cosec}\,\theta} = \frac{3a\sec^3\theta}{\sec^4\theta.\mathrm{cosec}\,\theta}$

$\rho = 3a\cos\theta.\sin\theta$

At $\theta = \frac{\pi}{4}$, $\langle \rho = 3a/2 \rangle$

→ S.T the Radius of curvature of the curve $x = a(\cos t + t\sin t)$ $y = a(\sin t - t\cos t)$ is 'at'.

$x = a(\cos t + t\sin t)$

$\frac{dx}{dt} = a(-\cancel{\sin t} + t\cos t + \cancel{\sin t})$

$y = a(\sin t - t\cos t)$

$\frac{dy}{dt} = a(\cancel{\cos t} + t\sin t - \cancel{\cos t})$

$\frac{dx}{dt} = at\cos t$, $\frac{dy}{dt} = at\sin t$

$y_1 = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$, $y_2 = \sec^2 t.\frac{dt}{dx} = \frac{\sec^2 t}{at\cos t}$

$y_2 = \frac{\sec^3 t}{at}$

$\rho = \frac{(1+\tan^2 t)^{3/2}at}{\sec^3 t}$

$\rho = \frac{\sec^3 t}{\sec^3 t}.at \Rightarrow \langle \rho = at \rangle$